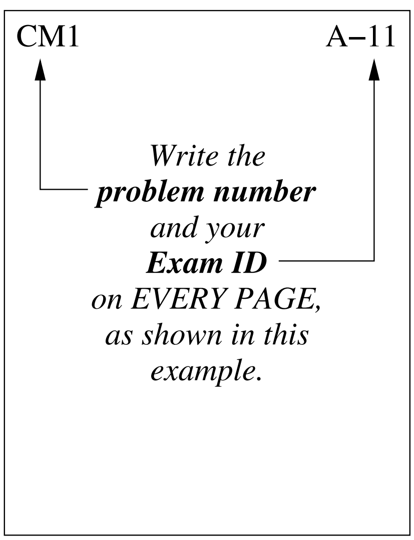
Department of Physics
Montana State University

Qualifying Exam
January, 2023

Day 1
Classical Mechanics



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(CM1) A particle of mass m moves without friction in the plane (polar coordinates r,ϕ) under the influence of a central potential $V(r)$.

(a) Find the Euler-Lagrange equations of motion for coordinates r and ϕ, subject to the unknown central potential $V(r)$. Indicate all conserved quantities in this problem.

(b) From time $t = 0$, with initial conditions $r = a$, $\phi = 0$, and $\frac{\partial \phi}{\partial t} = \omega$, the particle's trajectory is observed to follow a logarithmic spiral,

$$r = a\, e^{-\gamma \phi},$$

where γ is a constant. Use these facts to infer the potential $V(r)$.

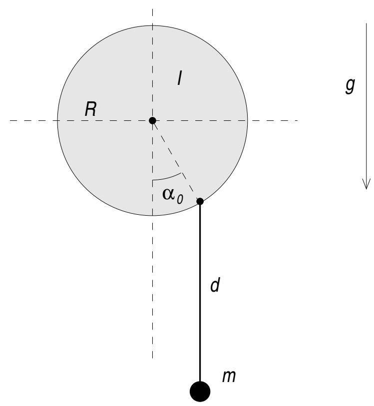
(CM2) A uniform circular disk of radius R can rotate about the axis perpendicular to the disk's plane and going through its center, with the moment of inertia about this axis being I. A rigid, massless rod of length d is attached to disk's circumference, with a mass particle m at its end. The pendulum can swing left-right around its suspension point.

At time $t = 0$ the disk is rotated such that pendulum attachment point is at angle α_0, and pendulum itself is vertical, and everything is motionless. Find the rotation angle of the disk $\alpha(t)$ after the system is let go.

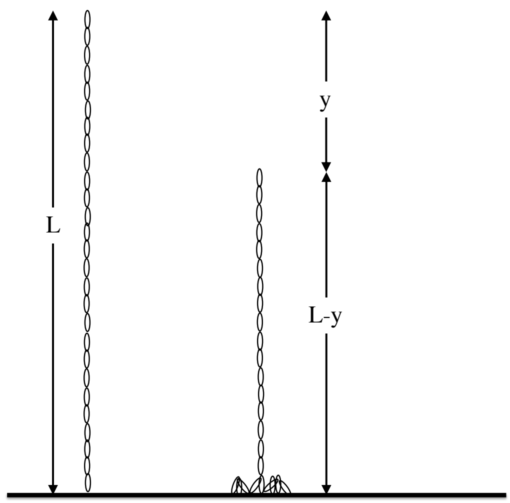
Use the following ratios to get good numbers,

$$\frac{d}{R} = 2 \, , \qquad \frac{I}{mR^2} = 2$$

and assume small amplitudes of the motion (as soon as possible in your solution to simplify algebra).



(CM3) A chain of length L and total mass M is released from rest with its lower end *just barely* touching the top of a table. Find the *total* force exerted by the table on the chain after it has fallen through a distance y. Assume that each link instantly comes to rest as it reaches the table, i.e. collisions of links with the table are completely inelastic, and treat the chain's mass as uniformly distributed along its length.



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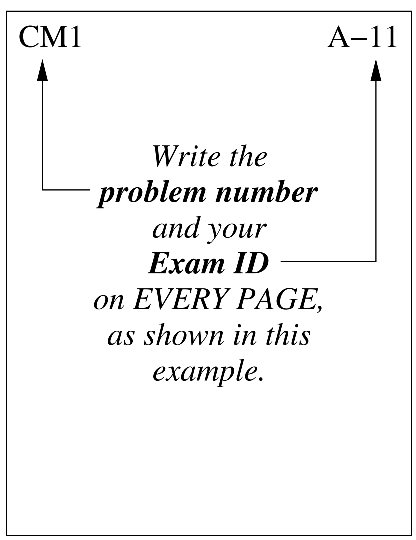
Day 2
Quantum Mechanics



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SPHERICAL HARMONICS $Y_{\ell,m}(\theta,\phi)$

Orthonormality of spherical harmonics

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \, Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

First several spherical harmonics explicitly

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}$$

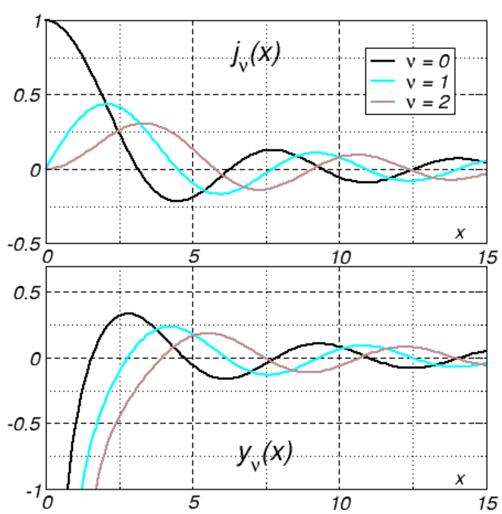
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta\ , \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)\ , \qquad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\sin\theta\cos\theta\ , \qquad Y_{2,\pm2} = \mp\sqrt{\frac{15}{32\pi}}e^{\pm i2\phi}\sin^2\theta$$

SPHERICAL BESSEL FUNCTIONS $j_\nu(x), y_\nu(x)$

They are solution to the equation

$$f''(x) + \frac{2}{x}f'(x) + \left(1 - \frac{\nu(\nu+1)}{x^2}\right) f(x) = 0$$



First few spherical functions of first and second kind:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$y_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

The first few zeros of the spherical Bessel functions of the first kind: $j_\nu(z_{\nu k}) = 0$

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$\nu = 0$	3.142	6.283	9.425	12.566
$\nu = 1$	4.493	7.725	10.904	14.066
$\nu = 2$	5.763	9.095	12.323	15.515

Orthogonality of spherical Bessel functions:

$$\int_0^1 x^2 dx \; j_\nu(z_{\nu n} x) j_\nu(z_{\nu m} x) = \delta_{nm} \frac{1}{2} j_{\nu+1}(z_{\nu n})^2$$

(QM1) A particle with mass m is in a one dimensional potential, $V(x)$. The spatial dependence of the ground state wave function is:

$$\psi_0(x) = Ae^{-k|x|}$$

(a) Determine the normalization constant, A.

(b) Find the potential, $V(x)$. Confirm that the potential has the correct units.

(c) Calculate the probability that the particle is within a distance d of the origin *and* determine how this probability changes if you consider the ground state of a particle in the same potential, but with half of the mass.

(QM2) A rotational state of a large molecule with total angular momentum quantum number $j \gg 1$ is represented by

$$|\psi\rangle = \frac{1}{\sqrt{3}}\,|j, m+1\rangle + \frac{1}{\sqrt{3}}\,|j, m\rangle + \frac{1}{\sqrt{3}}\,|j, m-1\rangle$$

with $-j < m < j$ being the quantum number of z-axis projection of the angular momentum.

(a) What are the probabilities of different outcomes for measurement of J_z in this state?

(b) Find the expectation value of measurement J_z in this state, and uncertainty of its measurement.

(c) Using properties of $J_\pm = J_x \pm iJ_y$ operators show that the expectation value of J_y in this state is zero. Find the expectation value of J_x in this state, simplify it in the case $j \pm m \gg 1$.

(d) Using previous two steps and the general uncertainty principle, what is the minimal uncertainty in measuring J_y in this state?

Recall the action of operator J_+ on a ket,

$$J_+\,|j, m\rangle = \sqrt{(j-m)(j+m+1)}\,|j, m+1\rangle$$

and use this to find action of J_-, if needed.

(QM3) Consider a particle with mass m confined in a 3D spherical infinite potential well with radius a:

$$V(r) = \begin{cases} 0 & r < a \\ \infty & \text{otherwise} \end{cases}$$

The system is perturbed by attractive delta potential at $(x, y, z) = (0, 0, a/2)$:

$$V'(\vec{r}) = -\alpha\delta(x)\delta(y)\delta(z - a/2)$$

where α is a positive constant.

(a) Up to normalization constants, determine the energies, wave functions, and degeneracies of the ground and first excited states of the *unperturbed* system (i.e., when $\alpha = 0$).

(b) Determine the normalization constant for the *unperturbed* ground state.

(c) Determine the energy (just the energy!) of the ground state of the *perturbed* system to first order in α. *Note:* integrals with the delta function are most easily computed in Cartesian coordinates.

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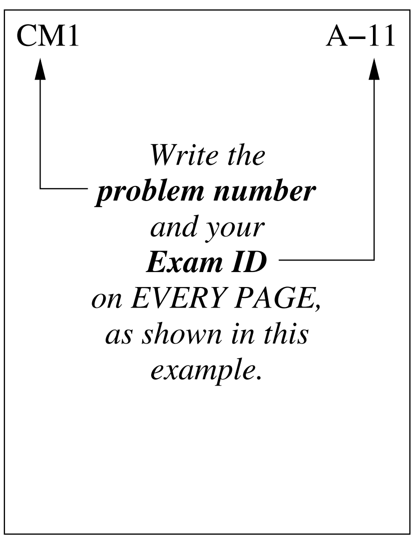
Day 3
Electricity and Magnetism



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(EM1) A conductive sphere of radius R is covered with a linear dielectric material of thickness b and dielectric constant $\epsilon_r \equiv \epsilon/\epsilon_0 \equiv 1 + \chi_e$, where ϵ is the electric permittivity and χ_e is the dielectric susceptibility. The electric potential $V \to 0$ as $r \to \infty$.

(a) Find the capacitance of the conducting sphere.

(b) Test your result in the limits $\epsilon_r \to 1$ and $\epsilon_r \to \infty$. What is the physical significance of each?

(EM2) A light of frequency ω from a distant source propagates through homogeneous interstellar medium with effective dielectric constant

$$\tilde{\varepsilon} = \varepsilon(1 + i\delta)$$

where ε, δ are real numbers, $\delta \ll 1$, and $i = \sqrt{-1}$. Assuming one-dimensional propagation, find the initial intensity of the light I_0 if we measure intensity I after it traverses distance D. Use provided information to make appropriate simplifications.

(EM3) A circular ring of radius a is placed in the xy plane. Total charge Q is uniformly distributed along the ring. The ring rotates at constant angular velocity ω about the z-axis that crosses its center.

(a) Qualitatively sketch the magnetic field everywhere in the xz-plane. Your sketch should reflect the direction and relative magnitude of the field at a few locations close to and far from the ring.

(b) Find the magnitude and direction of the field at far distance $r \gg a$ on the z-axis and in the xy plane, respectively.

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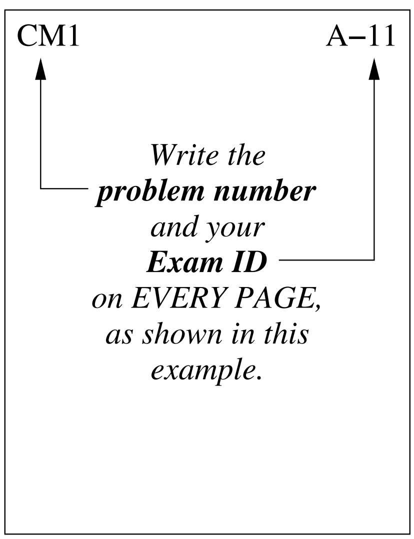
Day 4
Statistical and Thermal Physics



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(ST1) Two thin square plates, $a \times a$, face each other across an evacuated gap, distance $d \ll a$. Assume that the surfaces absorb all the radiation that is incident upon them. The plates are held at temperatures T and $T + \Delta T$.

(a) To lowest order in ΔT, find the net power transferred across the gap.

(b) Suppose now that another thin plate, with the same properties, is inserted between the first two. Assume that the outer two plates are held at their original temperatures, and that the system reaches a steady state. Find the temperature of the inserted plate and the net power transferred through the system.

(c) If the setup consists of N plates total (including the first two), what is the net power transferred through the system?

(ST2) Two identical monatomic, ideal gases with the same number of particles N, the same pressure P, but with different temperatures T_1 and T_2 are in their own vessels with volumes V_1 and V_2. The vessels are separated from each other by a valve and the entire system is thermally isolated from the environment. The valve is suddenly opened. After equilibrium has been reached:

(a) Determine the final temperature of the system.

(b) Find the change in entropy of the system, expressing your answer in terms of N, the Boltzmann constant k, and the initial temperatures.

(ST3) A liquid of temperature T is kept in a uniform external magnetic field $B\hat{z}$. The liquid is made of identical molecules, each of them having only two possible values for the component of its magnetic moment along the direction of the external magnetic field, $\mu_z = (1, -1)\mu_0$, μ_0 being constant. You may ignore interactions between molecules.

(a) Find the mean magnetic moment $\bar{\mu}$ of the liquid.

(b) Make a qualitative graph to show the temperature dependence of $\bar{\mu}$.

(c) Derive the approximate expressions for $\bar{\mu}$ at very high temperatures and very low temperatures.

(d) Find the specific heat C_B of the liquid, and discuss the temperature dependence of C_B.