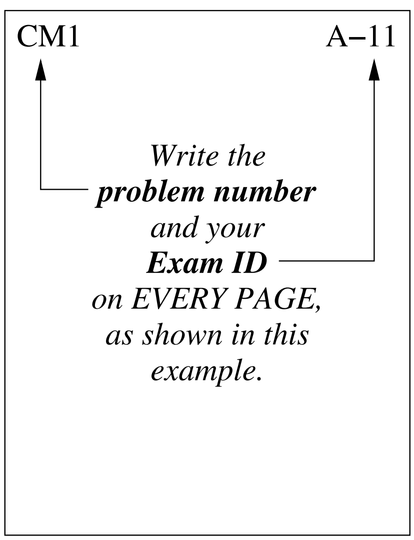
Department of Physics
Montana State University

Qualifying Exam
August, 2022

Day 1
Classical Mechanics



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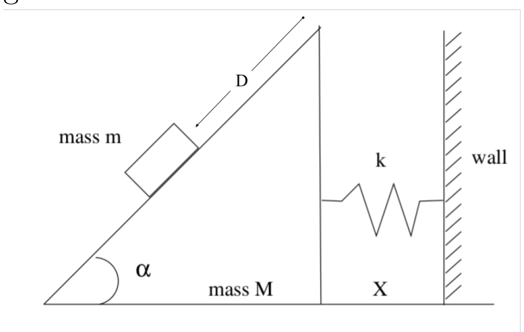
(CM1) A block of mass m slides with no friction down a ramp of mass M and height L under the force of gravity. The ramp is attached to the wall by a spring with spring constant k.



(a) Write the Lagrangian of the system in terms of X, the distance of the ramp from the wall, and of D, the distance of the object from the top of the ramp.

(b) Write the coupled equations of motion for these generalized coordinates.

(c) Suppose that α is small, indeed take it to zero for the purpose of this calculation. For $\alpha = 0$, find the normal frequencies and the normal modes.

(d) Describe the motion resulting from the two normal modes.

Solution:

a) First construct the Lagrangian with the given coordinates:

$$L = T - U =$$
$$= \frac{1}{2}M\dot{X}^2 + \underbrace{\frac{1}{2}m(\dot{X} + \dot{D}\cos\alpha)^2}_{\text{horizontal component}} + \underbrace{\frac{1}{2}m\dot{D}^2\sin^2\alpha}_{\text{vertical component}} - \frac{1}{2}k(X-l)^2 + mgD\sin\alpha$$

where l is the spring's relaxed length. We will take it to be zero for the remainder of the problem, as it does not qualitatively affect the final result of the problem.

b) $\frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{D}\cos\alpha$

$\frac{\partial L}{\partial X} = -kX$

$\frac{\partial L}{\partial \dot{D}} = m\dot{D} + m\dot{X}\cos\alpha$

$\frac{\partial L}{\partial D} = mg\sin\alpha$

The equations of motion are:

$(M+m)\ddot{X} + m\ddot{D}\cos\alpha + kX = 0$

$m\ddot{D} + m\ddot{X}\cos\alpha - mg\sin\alpha = 0$

c) Now simplify those equations of motion by assuming $\alpha = 0$:

$(M+m)\ddot{X} + m\ddot{D} + kX = 0$

$m\ddot{D} + m\ddot{X} = 0$

Assume $X = X_0\, e^{iwt}$ and $D = D_0\, e^{iwt}$, then

$$\begin{vmatrix} -(M+m)\omega^2 + k & -m\omega^2 \\ -m\omega^2 & -m\omega^2 \end{vmatrix} = 0$$

leading to

$-m\omega^2[-(M+m)\omega^2 + k] - m^2\omega^4 = 0.$

Solutions are $\omega = 0$ and $-(M+m)\omega^2 + k + m\omega^2 = -M\omega^2 + k = 0; \omega^2 = \frac{k}{M}$

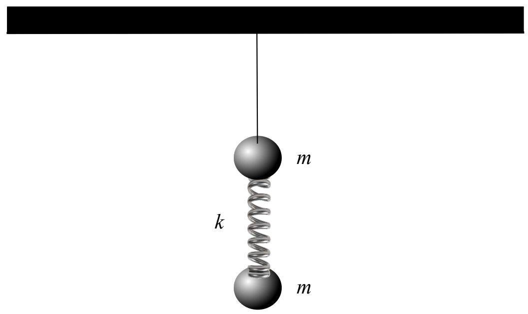
Then find the eigenvectors:

For $\omega = 0$: $X = 0$, $D = D_0 + D_1 t$

For $\omega^2 = \frac{k}{M}$: $X = X_0$, $D = -X_0$

d) As expected for horizontal slope, the first mode corresponds to a stationary mass M and mass m moving with uniform velocity, the second mode corresponds to oscillating mass M and stationary mass m.

(CM2) Two small balls of mass m are connected by a massless spring with a spring constant k, and the setup is attached to the roof by a massless string as shown in the figure. If the string is cut and the setup drops from rest, find the motion of the two balls.



Solution:

Analysis: the setup is only subject to gravity, and its center of mass has a free-fall motion. On top of that, the two balls' motion relative to the center of mass is an oscillation as governed by the spring. This is a conserved system, so we may use Lagrangian mechanics to find the equation of motion.

Approach: there are two degrees of freedom when the system is dropped from the vertical position as shown in the figure. We can use y_1 and y_2 to describe the height of the upper and lower balls relative to the floor, and constant l_0 the free length of the spring (when it is not stretched or compressed). The kinetic energy is

$$T = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2. \tag{1}$$

The potential energy includes two parts, the gravitational potential of the two masses, and the mechanical energy

$$V = mgy_1 + mgy_2 + \frac{1}{2}k(y_1 - y_2 - l_0)^2. \tag{2}$$

The Lagrangian is $L = T - V$, and for two independent variables, we derive two Euler-Lagrange equations

$$\begin{aligned} m\ddot{y}_1 &= -mg - k(y_1 - y_2 - l_0), \\ m\ddot{y}_2 &= -mg + k(y_1 - y_2 - l_0). \end{aligned} \tag{3}$$

It is easy to solve the equations by rewriting them as

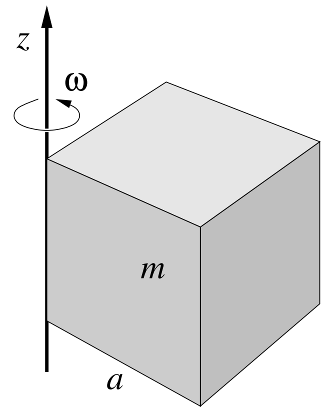
$$\begin{aligned} m(\ddot{y}_1 + \ddot{y}_2) &= -2mg, \\ m(\ddot{y}_1 - \ddot{y}_2) &= -2k(y_1 - y_2 - l_0). \end{aligned} \tag{4}$$

We define $Y_c = (y_1 + y_2)/2$, which is just the position of the center of mass, and $Y_d = (y_1 - y_2 - l_0)$, which is the change of the spring length relative to its free length. With this, we reduce the above differential equations to

$$\begin{aligned} \ddot{Y}_c &= -g, \\ m\ddot{Y}_d &= -2kY_d. \end{aligned} \tag{5}$$

The first equation describes the motion of the center-of-mass subject to gravity, and can be easily solved $Y_c = Y_{c0} - \frac{1}{2}gt^2$. The second equation describes an oscillation motion, the solution being $Y_d = Y_{d0}\cos(\omega t + \phi_0)$. The oscillation frequency is $\omega^2 = 2k/m$, and the other constants Y_{c0}, Y_{d0}, and ϕ_0 are determined by the initial state. In particular, it is easy to find $Y_{d0} = mg/k, \phi_0 = 0$ from the initial equilibrium. From there, the motion of each ball can be found as a superposition of free-fall and oscillation:

$$\begin{aligned} y_1 &= Y_c + \frac{1}{2}Y_d + \frac{1}{2}l_0 = Y_{c0} - \frac{1}{2}gt^2 + \frac{1}{2}\left[\frac{mg}{k}\cos(\omega t) + l_0\right], \\ y_2 &= Y_c - \frac{1}{2}Y_d - \frac{1}{2}l_0 = Y_{c0} - \frac{1}{2}gt^2 - \frac{1}{2}\left[\frac{mg}{k}\cos(\omega t) + l_0\right]. \end{aligned} \tag{6}$$



(CM3) A cube of mass m has constant density and edge length a. The cube rotates about a fixed axis (z) on one edge, at fixed angular frequency ω.

(a) Find the cube's angular momentum about the z-axis, and its kinetic energy.

(b) Calculate the (linear, instantaneous) velocity of the center of mass. What is the direction and magnitude of the force on the cube?

(c) The cube is released from its axis, so that there are no forces acting on it. Describe its subsequent angular and linear velocities. Compare the kinetic energy immediately before and after release. Do the same for angular momentum.

Solution:

(a) It is convenient to work in the lab frame, where the z-axis is stationary. The moment of inertia I about that axis is given by

$$dI = r^2\,dm = \rho r^2\,dV,$$

with density $\rho = m/a^3$. Radius r is measured perpendicular to the axis. The volume element is $dV = dx\,dy\,dz$.

$$I = \frac{m}{a^3}\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}\left(x^2+y^2\right)\,dx\,dy\,dz = \frac{2}{3}ma^2.$$

Note: We could have started from the moment of inertia I_0 about the center of mass. It just requires another step with the parallel axis theorem to get to I about the z-axis. The angular momentum and kinetic energy are:

$$\boxed{\mathbf{L} = I\omega\hat{\mathbf{z}} = \frac{2}{3}\omega ma^2\hat{\mathbf{z}}; \quad T = \frac{1}{2}I\omega^2 = \frac{1}{3}m(a\omega)^2.}$$

All of the energy is rotational because we worked in an inertial frame in which the axis of rotation is stationary.

(b) The center of mass of the cube is moving at velocity

$$\boxed{\mathbf{v} = \omega r_e\hat{\boldsymbol{\phi}} = \frac{\omega a}{\sqrt{2}}\hat{\boldsymbol{\phi}}.}$$

This is not a constant velocity, because $\hat{\phi}$ is not constant. The cube therefore experiences a centripetal acceleration of magnitude $r_e\omega^2$ directed inward (from the center of mass toward the z-axis, which we will label $-\hat{\mathbf{r}}_e$). The force is thus

$$\boxed{\mathbf{F} = -\frac{ma\omega^2}{\sqrt{2}}\hat{\mathbf{r}}_e.}$$

(c) When the cube is let go, Its center of mass continues in a straight line with velocity

$$\boxed{\mathbf{v} = \omega r_e\hat{\boldsymbol{x}} = \frac{\omega a}{\sqrt{2}}\hat{\boldsymbol{x}},}$$

in a direction tangent to its direction of travel at the moment of release. We've arbitrarily, but without loss of generality, named this direction $\hat{\boldsymbol{x}}$. We chose Cartesian unit vector because the direction $\hat{\boldsymbol{\phi}}$ varies with the azimuthal angle ϕ, which will continue to change as the cube moves away along a tangent to its original circular path. The cube rotates about its center of mass with unchanged angular velocity, $\boxed{\omega\hat{\mathbf{z}}.}$ The disappearance of force $\mathbf{F}$ does not result in any torque or work done, so **the kinetic energy and angular momentum about the z-axis remain the same as in part a.**

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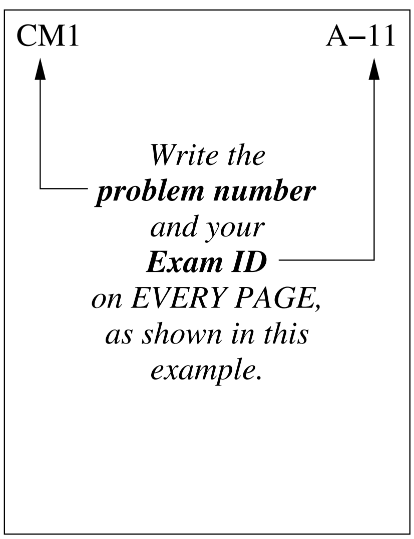
Day 2
Quantum Mechanics



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(QM1) A particle with spin $\hbar/2$, and magnetic moment $\boldsymbol{\mu} = \gamma \mathbf{S}$, where $\mathbf{S}$ is the spin operator and $\gamma > 0$ is the gyromagnetic ratio, sits in a uniform magnetic field that suddenly switches direction at $t = 0$:

$$\mathbf{B}(t) = \begin{cases} B_0 \hat{x}\,, & \text{for} \quad t < 0 \\ B_0 \hat{z}\,, & \text{for} \quad t > 0 \end{cases}$$

(a) Find the spinor wave function and the expectation value of the spin for $t < 0$ given that it is in the ground state of the system.

(b) Find the expectation value of the spin $\langle \mathbf{S}(t) \rangle$ for $t > 0$.

(c) Make a plot of the spin components $\langle S_i(t) \rangle$ as function of time.

Reminder: the spin-1/2 matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

Solution:

The Hamiltonian of spin in magnetic field is

$$\boxed{\mathcal{H} = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma \frac{\hbar}{2} \boldsymbol{\sigma} \cdot \mathbf{B}} \qquad \text{and we denote} \quad E_0 \equiv \frac{\gamma \hbar B_0}{2} \qquad \omega_0 \equiv \frac{\gamma B_0}{2}$$

(a) *Find the spinor wave function and the expectation value of the spin for $t < 0$ given that it is in the ground state of the system.*

Before $t = 0$ the system is in stationary state, that we know to be the ground state. The stationary state satisfies

$$E\psi = \mathcal{H}\psi = -E_0 \sigma_x \psi \qquad \Rightarrow \qquad E \begin{pmatrix} u \\ v \end{pmatrix} = -E_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The ground state corresponds to the lowest eigenvalue, with the spinor

$$\boxed{\text{ground state:} \qquad E_{GS} = -E_0\,, \qquad \psi_{GS} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The expectation value of the spin is

$$\langle \mathbf{S}(t < 0) \rangle = \psi_{GS}^{\dagger} \frac{\hbar}{2} \boldsymbol{\sigma} \psi_{GS} = \frac{\hbar}{2} \hat{x} + 0\hat{y} + 0\hat{z}$$

(b) *Find the expectation value of the spin* $\langle \mathbf{S}(t) \rangle$ *for* $t > 0$.

To find the expectation value of the spin after $t > 0$ we can use either the Schrödinger picture (time-evolved state) or the Heisenberg picture (time-evolved operator). In the Schrödinger picture we need to find the time-evolution of the eigenvectors of the Hamiltonian; and we have

$$i\hbar \frac{\partial}{\partial t}\psi = \mathcal{H}\psi = -E_0 \sigma_z \psi \qquad \Rightarrow \qquad \frac{\partial \psi}{\partial t} = i\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi$$

$$\Rightarrow \qquad \psi(t) = u \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0 t} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega_0 t} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\omega_0 t} \\ e^{-i\omega_0 t} \end{pmatrix}$$

where we set $u, v = 1/\sqrt{2}$, as they are determined from the initial condition $\psi(t=0) = \psi_{GS}$.

In the Heisenberg picture the state remains the same $\psi = \psi_{GS}$ but the operators evolve

$$i\hbar \frac{\partial}{\partial t} S_i = [S_i, \mathcal{H}] = -\gamma B_0 [S_i, S_z] = -\gamma B_0 \varepsilon_{izk} i\hbar S_k$$

$$\Rightarrow \qquad \frac{\partial}{\partial t} S_z = 0 \, , \qquad \frac{\partial}{\partial t} S_y = -2\omega_0 S_x \, , \qquad \frac{\partial}{\partial t} S_x = 2\omega_0 S_y \, ,$$

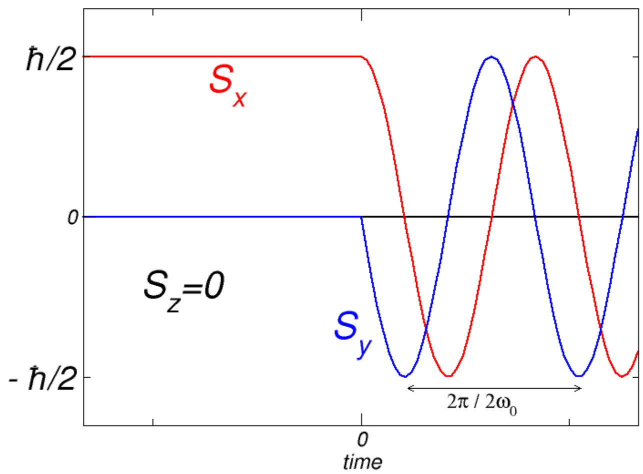
The solution is

$$S_x(t) = S_x(0)\cos 2\omega_0 t + S_y(0)\sin 2\omega_0 t \, , \qquad S_y(t) = S_y(0)\cos 2\omega_0 t - S_x(0)\sin 2\omega_0 t \, ,$$

and the expectation value is

$$\boxed{\langle \mathbf{S}(t > 0) \rangle = \psi(t)^\dagger \frac{\hbar}{2} \boldsymbol{\sigma} \psi(t) = \psi_{GS}^\dagger \mathbf{S}(t) \psi_{GS} = \hat{x}\frac{\hbar}{2}\cos 2\omega_0 t - \hat{y}\frac{\hbar}{2}\sin 2\omega_0 t + 0\hat{z}}$$

(c) *Make a plot of the spin components* $\langle S_i(t) \rangle$ *as function of time.*



ħ/2
0
- ħ/2
S_x
S_y
S_z=0
2π / 2ω_0
0
time

(QM2) A particle with mass m is confined to a 2D infinite square well with infinite potential barriers at $x = -a$, $x = a$, $y = -a$, and $y = a$.

The system experiences a weak potential that has the following form:

$$\hat{H}_1 = \beta xy$$

Here, β is a constant.

(a) Determine the energies, states, and degeneracies of the ground and first excited state of the unperturbed system.

(b) Treating the potential as a perturbation, determine the energies and states of the first two *excited* states of the perturbed system to lowest order in $\hat{H}_1$.

Note: The following integral will be useful:

$$\int_{-\pi/2}^{\pi/2} u \cos u \sin 2u \, du = 8/9$$

Solution:
(a) The unperturbed system is a 2D infinite square well. The solution to the time-independent Schrodinger equation is just the product of the solution for a 1D infinite potential well in the x direction and the solution for a 1D infinite potential well in the y direction. This solution is easily derived using separation of variables. Note that the the wells extend from $-a$ to a making the total width of each well $2a$.

The wavefunctions of the unpertubed system are:

$$\psi_{n_x,n_y} = \psi^{\text{1D}}_{n_x}(x)\psi^{1D}_{n_y}(y)$$

where $\psi^{1D}_{n'}(x)$ is the 1D solution for an infinite potential centered at $x = 0$ with a width of $2a$:

$$\psi^{1D}_{n'}(x) = \begin{cases} \sqrt{\frac{1}{a}} \cos\left(\frac{\pi n'}{2a}x\right), & n' = 1, 3, 5, ...(n' : \text{odd}) \\ \sqrt{\frac{1}{a}} \sin\left(\frac{\pi n'}{2a}x\right), & n' = 2, 4, 6, ...(n' : \text{even}) \end{cases}$$

The energies of these wavefunctions is the sum of the energies of the 1D systems (which again can be shown using the separation of variables):

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{8ma^2}(n_x^2 + n_y^2)$$

The **ground state** has the lowest energy, which corresponds to $n_x = n_y = 1$.

Quantum number: $n_x = n_y = 1$

State: $|n_x n_y\rangle = |11\rangle$

Wavefunction:

$$\langle\vec{r}|11\rangle = \psi_{1,1}(x,y) = \frac{1}{a}\cos\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2a}y\right)$$

Energy:

$$E_{1,1} = \frac{\hbar^2\pi^2}{4ma^2}$$

Degeneracy: 1

For the **first excited state**, one of the quantum numbers is 2, while the other is 1:

Quantum numbers: $n_x = 2$, $n_y = 1$ **or** $n_x = 1$, $n_y = 2$

States: $|21\rangle$ **or** $|12\rangle$

Wavefunctions:

$$\begin{aligned}\langle\vec{r}|21\rangle = \psi_{2,1}(x,y) &= \frac{1}{a}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{2a}y\right)\\ \langle\vec{r}|12\rangle = \psi_{1,2}(x,y) &= \frac{1}{a}\cos\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)\end{aligned}$$

Energy:

$$E_{2,1} = E_{1,2} = \frac{5\hbar^2\pi^2}{8ma^2}$$

Degeneracy: 2

(b) The perturbation will lift the degeneracy of the first excited state in the unperturbed system. To get the energies and states, degenerate perturbation theory is needed. Solve the eigenvalue equation for the perturbation over the the degenerate subspace. The eigenvalues are the first-order correction to the energies. The eigenvectors are the zeroth-order correction to the states (i.e., superpositions of the degenerate states that form the "good" states).

The matrix for the perturbation over the degenerate subspace is:

$$H_1 = \begin{pmatrix} \langle 12|\hat{H}_1|12\rangle & \langle 12|\hat{H}_1|21\rangle \\ \langle 21|\hat{H}_1|12\rangle & \langle 21|\hat{H}_1|21\rangle \end{pmatrix}$$

The matrix elements need to be evaluated using the unperturbed wavefunctions:

$$\begin{aligned} \langle 12|\hat{H}_1|12\rangle &= \int_{-a}^{a}\int_{-a}^{a} \frac{1}{a}\cos\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)(\beta xy)\frac{1}{a}\cos\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)dxdy \\ &= 0 \text{ (by symmetry)} \end{aligned}$$

Likewise,

$$\langle 21|\hat{H}_1|21\rangle = 0 \text{ (by symmetry)}$$

The off-diagonal elements are not 0:

$$\begin{aligned} \langle 12|\hat{H}_1|21\rangle &= \int_{-a}^{a}\int_{-a}^{a} \frac{1}{a}\cos\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)(\beta xy)\frac{1}{a}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{2a}y\right)dxdy \\ &= \frac{\beta}{a^2}\left[\int_{-a}^{a}\cos\left(\frac{\pi}{2a}x\right)x\sin\left(\frac{\pi}{a}x\right)dx\right]^2 \\ &= \frac{\beta}{a^2}\left[\left(\frac{2a}{\pi}\right)^2\int_{-\pi/2}^{\pi/2}\cos(u)u\sin(2u)du\right]^2 \\ &= \frac{\beta a^2 2^{10}}{81\pi^4} \end{aligned}$$

The remaining off-diagonal element is easily calculated from the results above because H_1 is Hermitian:

$$\begin{aligned}\langle 21|\hat{H}_1|12\rangle &= (\,\langle 12|\hat{H}_1|21\rangle)^* \\ &= \frac{\beta a^2 2^{10}}{81\pi^4}\end{aligned}$$

The matrix for the perturbation then is:

$$H_1 = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

where

$$\Delta = \frac{\beta a^2 2^{10}}{81\pi^4}$$

The first-order corrections to the energies are the eigenvalues of this matrix that can be found by solving the characteristic equation:

$$0 = \begin{vmatrix} -\epsilon & \Delta \\ \Delta & -\epsilon \end{vmatrix}$$

This equation yields the eigenvalues $\epsilon_- = -\Delta$ and $\epsilon_+ = \Delta$.

The zeroth-order correction to the states (aka the "good" states) are determined by the (normalized) eigenvectors which can be found by solving the eigenvalue equation for each eigenvalue:

$$H_1\vec{v}_\pm = \epsilon_\pm \vec{v}_\pm$$

The eigenvalue and normalized eigenvector pairs are then:

$$\begin{aligned}\epsilon_- = -\Delta = -\frac{\beta a^2 2^{10}}{81\pi^4}&: \quad \vec{v}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \epsilon_+ = \Delta = \frac{\beta a^2 2^{10}}{81\pi^4}&: \quad \vec{v}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

Translating the results from the degenerate subspace to the states of the unperturbed system. The first excited state will correspond to $\vec{v}_-$:

State:

$$|-\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)$$

Energy:

$$E_- = \frac{5\hbar^2\pi^2}{8ma^2} - \frac{\beta a^2 2^{10}}{81\pi^4}$$

The second excited state will correspond to $\vec{v}_+$:

State:

$$|+\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle)$$

Energy:

$$E_+ = \frac{5\hbar^2\pi^2}{8ma^2} + \frac{\beta a^2 2^{10}}{81\pi^4}$$

(QM3) Consider a quantum particle with mass m in a 1D harmonic potential of the form $V(x) = \frac{1}{2}\omega^2 x^2$.

(a) Determine how the product of the squares of the uncertainty of position and momentum ($\sigma_x^2 \sigma_p^2$) depend on the principle quantum number of the system, n. Find the algebraic dependence and make a plot for all quantum numbers up to $n = 2$.

(b) Of all possible energy eigenstates, which (if any) are states of minimum uncertainty (i.e. which energy eigenstates are at the limit established by the Heisenberg uncertainty principle)?

The following relationships using the ladder operators may be useful

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$$
$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$$
$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\hat{a^\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Solution:

(a) The square of the uncertainty for position and momentum can be expressed as a difference between two expectation values:

$$\sigma_x^2 = \left\langle x^2 \right\rangle - (\langle x \rangle)^2$$
$$\sigma_p^2 = \left\langle p^2 \right\rangle - (\langle p \rangle)^2$$

By symmetry (i.e., because $V(x)$ is symmetric around $x = 0$):

$$\langle x \rangle = 0$$
$$\langle p \rangle = 0$$

So, the problem simplifies to finding $\langle x^2 \rangle$ and $\langle p^2 \rangle$ for the quantum harmonic oscillator as a function of the principal quantum number, n. Determining this quantity is most easily accomplished using the provided ladder operators.

Using the provided relationships, $\hat{x}$ and $\hat{p}$ can be expressed as linear combinations of the ladder operators.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a^\dagger}\right)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a^\dagger} - \hat{a}\right)$$

Squaring the above expressions and noting that $\hat{a}$ and $\hat{a^\dagger}$ do not commute yields:

$$\hat{x^2} = \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \hat{a^\dagger}^2 + \hat{a^\dagger}\hat{a} + \hat{a}\hat{a^\dagger}\right)$$
$$\hat{p^2} = -\frac{\hbar m\omega}{2}\left(\hat{a}^2 + \hat{a^\dagger}^2 - \hat{a^\dagger}\hat{a} - \hat{a}\hat{a^\dagger}\right)$$

For $\langle x^2\rangle$ then (using the provided operations):

$$\begin{aligned}
\left\langle x^2\right\rangle &= \langle n|\hat{x^2}|n\rangle \\
&= \frac{\hbar}{2m\omega}\langle n|\left(\hat{a}^2 + \hat{a^\dagger}^2 + \hat{a^\dagger}\hat{a} + \hat{a}\hat{a^\dagger}\right)|n\rangle \\
&= \frac{\hbar}{2m\omega}(\sqrt{n}\sqrt{n-1}\,\langle n|n-2\rangle + \sqrt{n+1}\sqrt{n+2}\,\langle n|n+2\rangle \\
&\quad + \sqrt{n}\sqrt{n}\,\langle n|n\rangle + \sqrt{n+1}\sqrt{n+1}\,\langle n|n\rangle) \\
&= \frac{\hbar}{2m\omega}(2n+1)
\end{aligned}$$

Note that $\langle n|n-2\rangle = \langle n|n+2\rangle = 0$ because of orthogonality of the states.

Likewise,

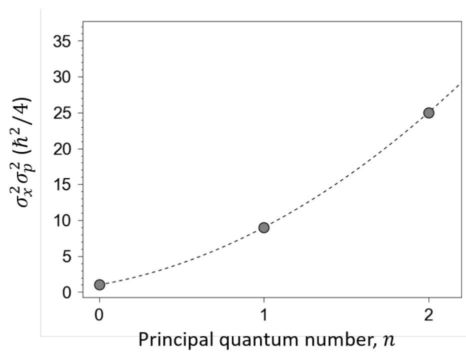
$$\begin{aligned}
\left\langle p^2\right\rangle &= \langle n|\hat{p^2}|n\rangle \\
&= -\frac{\hbar m\omega}{2}\langle n|\left(\hat{a}^2 + \hat{a^\dagger}^2 - \hat{a^\dagger}\hat{a} - \hat{a}\hat{a^\dagger}\right)|n\rangle \\
&\ldots \\
&= \frac{\hbar m\omega}{2}(2n+1)
\end{aligned}$$

Therefore, for the quantum harmonic oscillator, the product of $\sigma_x^2 \sigma_p^2$ increases quadratically with quantum number, n

$$\begin{aligned}\sigma_x^2 \sigma_p^2 &= \left\langle x^2 \right\rangle \left\langle p^2 \right\rangle \\ &= \frac{\hbar^2}{4}(2n+1)^2\end{aligned}$$

This value can be tabulated for the first few states of the QHO (the ground state corresponds to $n = 0$):

n	$\sigma_x^2 \sigma_p^2$
0 (ground state)	$\frac{\hbar^2}{4}(1)$
1	$\frac{\hbar^2}{4}(9)$
2	$\frac{\hbar^2}{4}(25)$
3	$\frac{\hbar^2}{4}(49)$
...	...

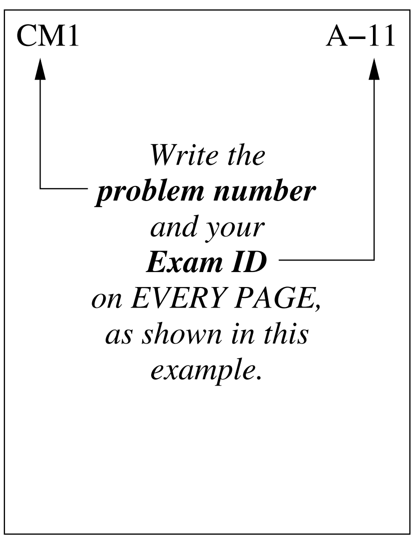


(b) The Heisenberg uncertainty principle states that $\sigma_x \sigma_p \geq \hbar/2$ which means the minimum possible value of $\sigma_x^2 \sigma_p^2$ is $\hbar^2/4$. The ground state of the quantum harmonic oscillator has this minimum value (no matter what the mass of the particle or the frequency of the potential are!).

Department of Physics
Montana State University

Qualifying Exam
August, 2022

Day 3
Electricity and Magnetism



- Show your work.
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Information

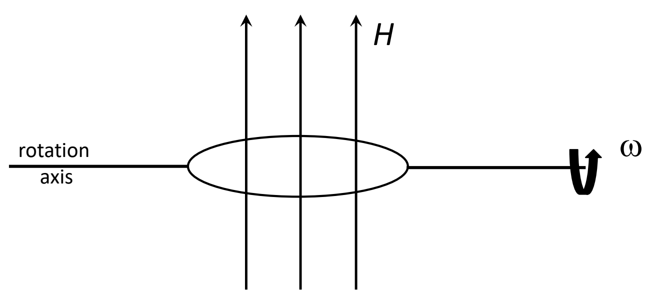
Useful vector identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

Table 1: PHYSICAL CONSTANTS

SYMBOL	NAME	VALUE	UNITS
c	speed of light in vacuum	299792458	m s^{-1}
G	gravitational constant	$6.67408 \cdot 10^{-11}$	N m^2 kg^{-2}
g	standard gravity	9.80665	m s^{-2}
h	Planck constant	$6.62607015 \cdot 10^{-34}$	J s
		$4.13566770 \cdot 10^{-15}$	eV s
$\hbar = h/2\pi$	reduced Planck constant	$1.05457182 \cdot 10^{-34}$	J s
		$6.58211957 \cdot 10^{-16}$	eV s
e	elementary charge	$1.602176634 \cdot 10^{-19}$	C
ε_0	electric constant	$8.854 \cdot 10^{-12}$	C^2 N^{-1} m^{-2}
μ_0	magnetic constant	$4\pi \cdot 10^{-7}$	T m A^{-1}
N_A	Avogadro's constant	$6.02214076 \cdot 10^{23}$	mol^{-1}
k_B	Boltzmann's constant	$1.380649 \cdot 10^{-23}$	J K^{-1}
$R = N_A k_B$	gas constant	8.314462618	J mol^{-1} K^{-1}
$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$	Stefan-Boltzmann constant	$5.670367 \cdot 10^{-8}$	W m^{-2} K^{-4}
m_e	electron mass	$9.109 \cdot 10^{-31}$	kg
		0.5109	MeV
m_p	proton mass	$1.672 \cdot 10^{-27}$	kg
		938.2	MeV
m_n	neutron mass	$1.674 \cdot 10^{-27}$	kg
		939.5	MeV

(EM1) A thin ring of copper with electrical conductivity σ, density ρ, mass m, and radius r rotates about an axis perpendicular to a uniform magnetic field H with initial angular frequency ω_0. Under the assumptions that the energy goes into Joule heating and ω_0 is rapid enough that the ring can complete many full rotations, find an expression for the time it takes for the angular frequency to decrease to ω_0/e (note $\ln(e) = 1$) in terms of the given quantities. Ignore the ring's self inductance.



Solution:
Solution:

(1) Using a conservation of energy approach: The emf should look like: $\epsilon = \pi r^2 H\omega \sin\omega t$.

The electrical power dissipated by the ring will be $P = I^2 R = \epsilon^2/R$, where the resistance R of the ring is $R = \frac{(2\pi r)^2 \rho}{\sigma m}$.

Putting this together and time averaging gives $P_{avg} = \frac{\sigma m r^2 H^2 \omega^2}{8\rho}$, for the electrical power dissipated in one rotation.

The electrical power dissipated shows up as a loss of mechanical energy. Therefore, energy conservation tells us that: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mr^2\frac{\omega^2}{2}\right) = -\frac{\sigma m r^2 H^2 \omega^2}{8\rho}$. This can be simplified to $\dot{\omega} = -\frac{\sigma H^2}{4\rho}\omega$.

This equation has the solution $\omega = \omega_o e^{-t/\tau}$ with $\tau = \frac{4\rho}{\sigma H^2}$, so our value for t is τ.

(2) Using the torque: $\tau = \frac{1}{2}mr^2\dot{\omega}$ and $\vec{\tau} = \vec{\mu} \times \vec{H}$, where μ is the dipole moment.

The current $I = \frac{H\pi r^2\omega}{R}\sin\omega t$, so the dipole moment for the loop is $\mu = \frac{H\pi^2 r^4\omega}{R}\sin\omega t$. This leads to a torque of magnitude $\tau = \frac{H^2\pi^2 r^4\omega}{R}\sin^2\omega t$.

Placing the equation for R from the first solution into the last expression gives the differential equation $\dot{\omega} = -\frac{\sigma H^2}{4\rho}\omega$, identical to the solution found above.

(EM2) Visible light is incident normally on an aluminum (Al) plate. Describe how the electric (and magnetic) fields behave inside a thick plate, and use this to estimate how thick should the plate be to reduce the transmitted light *power* to 10^{-6} of the incident power. (You can ignore all reflections inside the plate to estimate transmission.)

Take the visible light frequency to be $\nu = 5 \cdot 10^{14}$ Hz with wavelength $\lambda \sim 700$ nm. The conductivity of Al is $\sigma = 3.5 \cdot 10^7\ \Omega^{-1}\,\mathrm{m}^{-1}$ (SI units) $= 3 \cdot 10^{17}\ \mathrm{s}^{-1}$ (Gaussian, cgs units).

Solution:
The EM fields inside the metal cause flowing currents

$$\mathbf{j} = \sigma \mathbf{E}$$

and dissipation of energy, and thus decay exponentially over the skin-depth. We need to find this lengthscale.

In Gaussian system we write the Maxwell equations inside the metal (setting $\varepsilon = 1$, $\mu = 1$)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Assuming all fields and currents oscillate with the same frequency

$$(\mathbf{E},\ \mathbf{B},\ \mathbf{j}) \quad = (\mathbf{E}(x), \mathbf{B}(x), \mathbf{j}(x))\ e^{-i\omega t}$$

we write these equations for complex amplitudes

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}(x) &= \frac{i\omega}{c}\mathbf{B}(x) \\ \boldsymbol{\nabla} \times \mathbf{B}(x) &= \left(\frac{4\pi}{c}\sigma - \frac{i\omega}{c}\right)\mathbf{E}(x) \end{aligned}$$

we combine the two equations, using $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ in $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$,

$$-\nabla^2 \mathbf{E}(x) = \frac{i\omega}{c}\left(\frac{4\pi}{c}\sigma - \frac{i\omega}{c}\right)\mathbf{E}(x)$$

In vacuum we have the usual wave equation with traveling wave solution

$$-\nabla^2 \mathbf{E}(x) = \frac{\omega^2}{c^2}\mathbf{E}(x) \qquad \Rightarrow \qquad E(x) = E_0 e^{\pm ikx} \qquad with \qquad k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

Inside metal we have, noticing that $\sigma \gg \omega$ (easy to see in Gaussian units!)

$$-\nabla^2 E(x) = \frac{i\omega}{c}\left(\frac{4\pi}{c}\sigma - \frac{i\omega}{c}\right) E(x) \approx i\frac{4\pi\sigma\omega}{c^2} E(x)$$

Looking for solution $E(x) \propto e^{i\kappa x}$ we have condition on κ:

$$\kappa^2 = i\frac{4\pi\sigma\omega}{c^2}$$

which means the wave number inside the metal is a complex number:

$$\kappa = \sqrt{i\frac{4\pi\sigma\omega}{c^2}} = \sqrt{i}\sqrt{\frac{4\pi\sigma\omega}{c^2}} = \pm\frac{1+i}{\sqrt{2}}\sqrt{\frac{4\pi\sigma\omega}{c^2}} = (1+i)\frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}$$

where we selected only the + solution that decays into the metal:

$$\boxed{E(x) = E_0 e^{ik'x - x/\delta}\,, \qquad k' = k\sqrt{\frac{2\pi\sigma}{\omega}}\,, \qquad \delta^{-1} = k' = k\sqrt{\frac{2\pi\sigma}{\omega}} = k\sqrt{\frac{\sigma}{\nu}}}$$

The decay length is

$$\delta = \frac{\lambda}{2\pi}\sqrt{\frac{\nu}{\sigma}} = \frac{700\ nm}{2\pi}\sqrt{\frac{5\cdot 10^{14}}{3\cdot 10^{17}}} \sim 4\ nm$$

The light power flux at some position x is given by the time-averaged Poynting vector $\mathbf{S}(x) = \frac{c}{8\pi}\mathbf{E}(x)\times\mathbf{H}^*(x)$ and is proportional to the square of the field's amplitude so the reduction of transmission over the plate is equal to the ratio of the incoming and outgoing amplitudes (and here we neglect the small amount of reflection back into the plate that will result in multiple internal reflections)

$$\left.\frac{|E(x)|^2}{E_0^2}\right|_{x=L} = e^{-2L/\delta} \sim 10^{-6} \qquad \Rightarrow \qquad L \sim \frac{\delta}{2}\ln 10^6 \sim 30\ nm$$

Note 1: At the interfaces we have mismatch of impedances that lead to additional reduction in the transmission amplitudes. For example, at the first vacuum-metal interface of the thick plate we have

$$\frac{E_T}{E} = \frac{2}{1+\kappa/k}$$

but this leads to only about $|\kappa/k| \sim \sqrt{\sigma/\nu} \sim 10$ times amplitude reduction, and the main drop in the energy flux still comes from the exponential decay inside the metal.

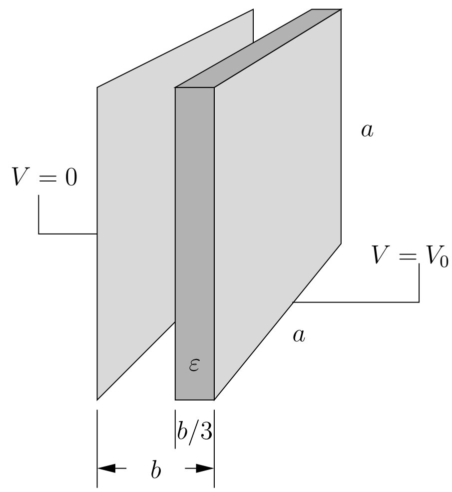
Note 2: In SI units the system of equations for complex amplitude is

$$\begin{aligned} \nabla \times \mathbf{E}(x) &= i\omega \mathbf{B}(x) \\ \nabla \times \mathbf{B}(x) &= \mu_0 \left(\sigma - \varepsilon_0 i\omega\right) \mathbf{E}(x) \end{aligned}$$

and the skin depth formula in the limit $\sigma \gg \omega\varepsilon_0$ is

$$-\Delta\mathbf{E} = i\omega\mu_0\sigma\mathbf{E} \qquad \Rightarrow \qquad \kappa = \frac{1+i}{\sqrt{2}}\sqrt{\mu\sigma\varepsilon_0} \qquad \Rightarrow \qquad \delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

(EM3) Two thin square, parallel metal plates, $a \times a$, are separated by a gap $b \ll a$. The gap is 1/3 filled by dielectric, permittivity ε, attached to one plate as shown. The plate adjacent to the dielectric is at potential V_0, and the other is grounded. Find the electric field between the plates, ignoring edge effects. Also find the bound and free charge densities.



Solution:
Let's define x as the coordinate perpendicular to the plates, with $x = 0$ on the grounded plate, and $x = b$ and the other plate. Since $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$ (free charge only), and because of the slab symmetry, we infer that $\mathbf{D} = D\hat{x}$, with D a constant, both in the dielectric and in the empty space between the plates. We know that $\mathbf{D} = \epsilon \mathbf{E}$, so

$$\boxed{\mathbf{E} = \begin{cases} \dfrac{D\hat{\mathbf{x}}}{\epsilon_0}, & 0 < x < \dfrac{2b}{3}; \\ \dfrac{D\hat{\mathbf{x}}}{\epsilon}, & \dfrac{2b}{3} \leq x < b. \end{cases}}$$

We need to put D in terms of the given applied voltage. Using $\mathbf{E} = -\nabla V$,

$$-V_0 = \int_0^b E_x \, dx = \frac{bD}{3}\left(\frac{2}{\epsilon_0} + \frac{1}{\epsilon}\right) \implies \boxed{D = \frac{-3V_0 \epsilon \epsilon_0}{b(2\epsilon + \epsilon_0)}}.$$

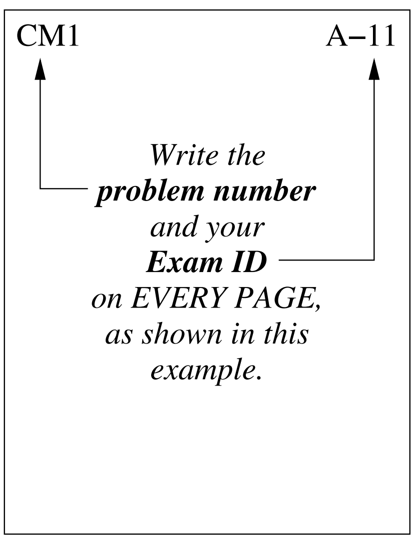
The electric field and the displacement are zero outside the plates. Employing both forms of Gauss's Law ($\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$ and $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$), we find the surface charges to be:

charge↓ location→	$x = 0$	$x = 2b/3$	$x = b$
σ	D	$-D(1 - \epsilon_0/\epsilon)$	$-D\epsilon_0/\epsilon$
σ_f	D	0	$-D$
$\sigma_b = \sigma - \sigma_f$	0	$-D(1 - \epsilon_0/\epsilon)$	$D(1 - \epsilon_0/\epsilon)$

Department of Physics
Montana State University

Qualifying Exam
August, 2022

Day 4
Statistical and Thermal Physics



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(ST1) The Sun has a surface temperature of $T_S = 5800$ K and its radius is $R_S = 7.0\times10^8$ m. The Earth has a radius of $R_E = 6.4 \times10^6$ m and it is a mean distance $d = 1.5\times10^{11}$ m from the sun. Assume that both the Sun and the Earth absorb all electromagnetic radiation that is incident upon them, and that the Earth's temperature T_E is in a steady state, so that it does not change with time. Find an expression for T_E and provide an approximate numerical value.

Solution:
The sun radiates energy uniformly in all directions with a rate $P = \sigma T_S^4 4\pi R_S^2$. Note σ is the Stefan-Boltzmann constant.

A disc of area πR_E^2 intercepts light from the sun, so the rate with which energy strikes the Earth's bright side is $P_{EA} = P\frac{\pi R_E^2}{4\pi d^2}$.

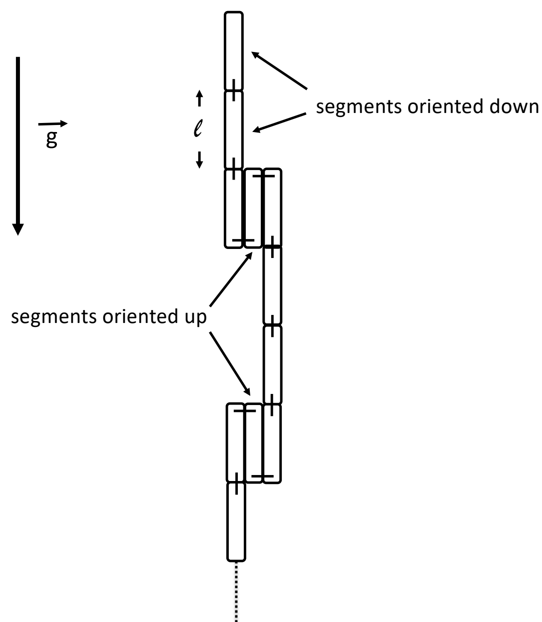
The Earth absorbs, but also emits radiation. If the two are in equilibrium, Earth's temperature stays constant. Thus if $\sigma T_S^4 4\pi R_S^2 \frac{R_E^2}{4d^2} = \sigma T_E^4 4\pi R_E^2$, this is the case. This reduces to

$$T_E = T_S \left(\frac{R_S}{2d}\right)^{1/2} .$$

Notice how it is independent of the Earth's radius - any size object at this distance from the sun will have the same temperature! Substituting numbers, after basic cancellations we get

$$T_E = 580\sqrt{\frac{7}{30}} = 580\sqrt{\frac{1}{4(1+2/28)}} \approx 290\left(1-\frac{1}{28}\right) \approx 280\ K$$

(ST2) A rubber band can be modeled as a chain of N segments of identical length ℓ joined end to end. Imagine that a weight W hangs from the very end of our rubber band while the other end is firmly fastened to a peg anchored to the wall. The temperature is T. Each segment can be in one of two states, down or up, relative to the vertical. The figure illustrates a short length of the rubber band with segments oriented up and down. Determine the average length $\bar{L}$ of the rubber band as a function of W and T. Show that your model agrees with your physical expectations for $\bar{L}$ at low and high T.



Solution:
There are two possible states for each segment. If the segment is down, the potential energy of the weight is decreased by $-W\ell$. If the segment is up, the potential energy of the weight is increased by $+W\ell$. The partition function for the system is $Z = e^{W\ell\beta} + e^{-W\ell\beta}$, where $\beta = 1/\mathrm{k}T$. The average contribution of each segment to the total average length $\bar{L}$ is

$$\bar{\ell} = \frac{\partial}{\partial(W\beta)} \ln Z \quad = \quad \ell \frac{e^{W\ell\beta} - e^{-W\ell\beta}}{e^{W\ell\beta} + e^{-W\ell\beta}}$$

In turn, $\bar{L} = N\ell \frac{e^{W\ell\beta} - e^{-W\ell\beta}}{e^{W\ell\beta} + e^{-W\ell\beta}}$.

At high T, one can expand the exponentials to find $\bar{L} \sim (N\ell)W\ell\beta \to 0$. This tells us that all the links are in a sense 'coiled up' and the rubber band shrinks.

At low T there is inadequate thermal energy to populate the 'up' state so $\bar{L} \sim (N\ell)$ - the band is completely extended.

(ST3) A sealed can of volume V_1 containing air at room temperature ($T_1 = 300\,\mathrm{K}$) is run over and flattened by a large, fast moving truck. Amazingly, the can does not leak! At the moment the truck is on top of the can, its volume is $V_2 \approx V_1/32$. Estimate the temperature, T_2, of the air in the can at that moment. Since air is an ideal gas and is mostly diatomic, you may assume $C_P = 7R/2$ and $C_V = 5R/2$.

Solution:
Because it happens fast, the compression is adiabatic. The adiabatic process is described by

$$P_1 V_1^\gamma = P_2 V_2^\gamma, \quad \text{where} \quad \gamma = \frac{C_P}{C_V} = \frac{7}{5}.$$

The ideal gas law, $PV = nRT$, allows us to eliminate the pressure, so

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 32^{2/5} = 4.$$

Hence,

$$\boxed{T_2 = 1200\,\mathrm{K}.}$$