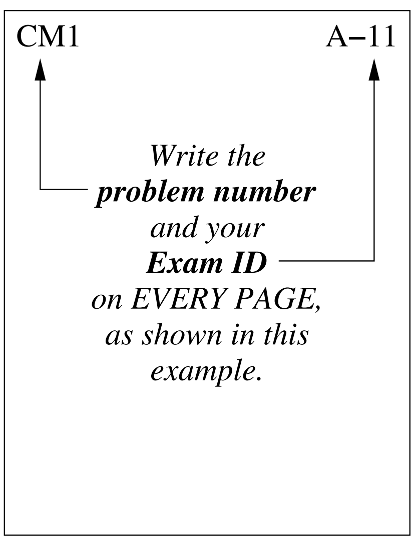
Department of Physics
Montana State University

Qualifying Exam
August, 2024

Day 1
Classical Mechanics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(CM1) A harmonic oscillator has natural frequency ω_0 and a weak damping $0 < \gamma \ll \omega_0$. It is driven by a force at a single frequency, with $F(t) = \omega_0^2 f_0 e^{i\Omega t}$, with f_0 a constant. The equation of motion reads

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \omega_0^2 f_0 e^{i\Omega t}.$$

[Here we work with a complex x for convenience. All the frequencies $(\omega_0, \Omega, \gamma)$ are real. To get physical solutions, we can simply take the real part of x. You may or may not need Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$.]

(a). When $f_0 = 0$ (i.e., without external drive), find a general expression for $x(t)$ for arbitrary, non-trivial initial conditions.

(b). For non-zero f_0, show that a solution of the form $x(t) = B\exp(i\Omega t)$ solves the equation. Express the complex amplitude B in terms of other parameters of the problem.

(c). Show that the sum of the solutions from (a) and (b) also solves the equation. Suppose we wait long enough, which term in the sum dominates?

(d). Consider only the dominant term from (c). Define the response function $T(\Omega) = x/f_0$. Sketch out the shape of $|T(\Omega)|$ as a function of $\Omega > 0$. When will $|T(\Omega)|$ achieve its maximum and what is its value there? Make sure you also include the correct asymptotic behaviors in the limit $\Omega \ll \omega_0$ and $\Omega \gg \omega_0$. In the two limits, is $x(t)$ in phase or out of phase with $F(t)$?

(CM2) As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Figure 1, with two identical atoms each of mass m connected by two identical springs (with spring constant K and rest length l_0) to a single atom of mass M. Assume that all motion is in the x direction and use the positions of the masses x_1, x_2 and x_3, as generalized coordinates.

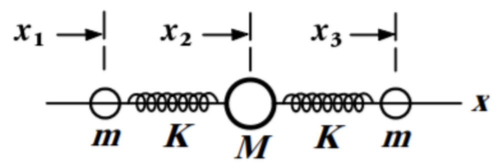


Figure 1: Cartoon of a CO2 molecule.

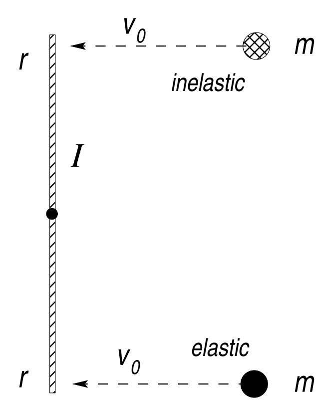
(a). Write down the Lagrangian and find the equations of motion.

(b). Find the normal frequencies of the system. Assuming $m \simeq M$, rank the magnitude of the eigenfrequencies from the highest to the lowest. Show that one of the normal frequencies is zero.

(c). Find the normal mode vectors. (You need not normalize them.) Describe in words the motion corresponding to each nomral mode.

(CM3) Two balls hit a rod that can freely pivot around its center. Both balls have the same mass m and speed v_0. They hit perpendicular to the rod at equal distances r from the pivot axis. One ball undergoes a perfectly elastic collision with the rod and the other a perfectly inelastic (i.e. it sticks to the rod). Assume each collision happens instantaneously, but the elastic collision happens just before the inelastic. The time between the collisions is very short, so that the rod does not change its position significantly during that interval, and both balls hit the rod at 90 degree angles.

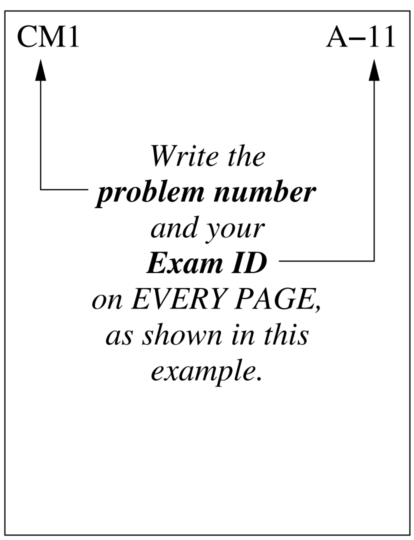
What will be the eventual direction and angular speed of rotation of the rod and stuck ball? The moment of inertia of the rod about its pivot is $I \gg mr^2$.



Department of Physics
Montana State University

Qualifying Exam
August, 2024

Day 2
Quantum Mechanics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(QM1) A particle of mass m is in a 2d $a \times a$ square well potential with infinite walls at $x = 0$, $x = a$, $y = 0$, $y = a$. The particle is subject to a perturbation

$$V'(x,y) \;=\; \alpha\, \delta(x-y) \quad , \tag{1}$$

where $\alpha > 0$ is real. Find **all** the states in the **first excited energy level** of the **unperturbed** system. Use degenerate perturbation theory to compute the first order corrections to the energies of the first excited state.

Some or all of the following integrals may prove helpful $(m, n = 1, 2, 3, \cdots)$

$$\int_0^1 \sin^2(n\pi t)\, dt \;=\; \int_0^1 \cos^2(n\pi t)\, dt \;=\; \frac{1}{2}$$

$$\int_0^1 \sin^2(n\pi t)\, \sin^2(m\pi t)\, dt = \frac{1}{4} + \frac{1}{8}\delta_{nm}$$

(QM2) A particle of mass m is bound in a harmonic 1D potential of the form:

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

At time $t = 0$, the particle is in the following state:

$$|\Psi(0)\rangle = A\Big(|0\rangle + 2\,|2\rangle\Big)$$

where A is a constant and $|n\rangle$ is the n^{th} energy eigenstate.

a. Find $|\Psi(t)\rangle$ for all times $t \geq 0$.

b. Determine the expectation value of the position ($\langle x\rangle$) and momentum ($\langle p\rangle$) of the particle for all times t.

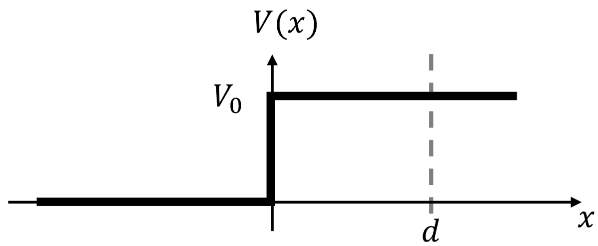
c. At time $t = T$, a measurement of the particle energy is made. Determine all possible outcomes of the measurement and their probabilities.

d. For the same particle in (c), assume that the measurement outcome is the largest of the the possible outcomes. What will the the possible measurement outcomes for the total energy of the particle for $t > T$?

Reminder: raising and lowering operators for harmonic potential are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}\ , \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$$

(QM3) A beam of particles (i.e. a plane wave state) is traveling in the $+x$ direction from $x = -\infty$ with energy E and mass m. The beam of particles is incident on the step potential of height V_0 where $V_0 = 2E$. The wave functions cannot be normalized, so assume the incident plane wave has an amplitude A.



a. Determine the wave function for all x that describes the effect of the step potential on the incident beam.

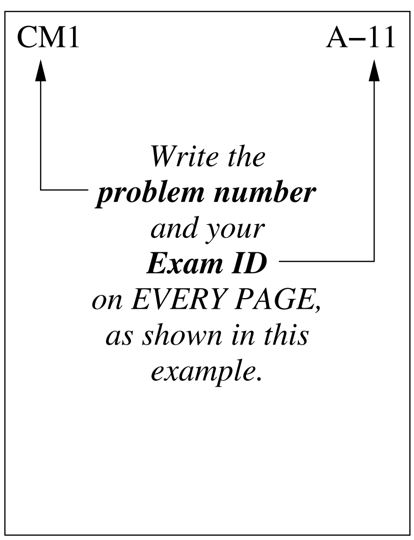
b. Determine the position, d at which the probability density of detecting particles decreases by a factor of 10 from the probability density at $x = 0$.

c. Determine the reflection coefficient of the potential. Justify your answer.

Department of Physics
Montana State University

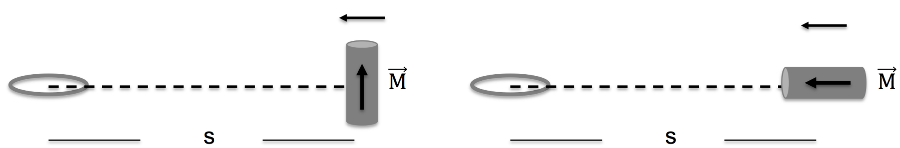
Qualifying Exam
August, 2024

Day 3
Electricity and Magnetism



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(EM1) A circular cylinder bar magnet of radius a, length l, and uniform **magnetization** $\vec{M}$ is pushed toward a circular loop of radius b and self-inductance L. The magnet remains at distances s far from the loop: $s \gg a, b, l$. The loop is a perfect conductor. Initially, the magnet is at $s \to \infty$ and there is no current in the loop.



(a) (b)

a. In the configuration shown in Figure (a), find the magnetic flux through the loop when the bar magnet is at the distance s ($s \gg a, b, l$).

b. In the configuration shown in Figure (a), what is the direction of the induced current in the loop when the bar magnet moves toward the circular loop?

c. Find the magnitude of the induced current as a function of s, when the bar magnet is moving towards the loop from infinity.

d. What is the current in the loop if the orientation of the bar magnet is different, as shown in Figure (b)?

(EM2) A spherical shell surrounds a central point charge at the origin. The graph represents the electric field as a function of distance from the origin. From the graph determine (and argue your answers):

(a) The approximate value of the point charge at the origin;

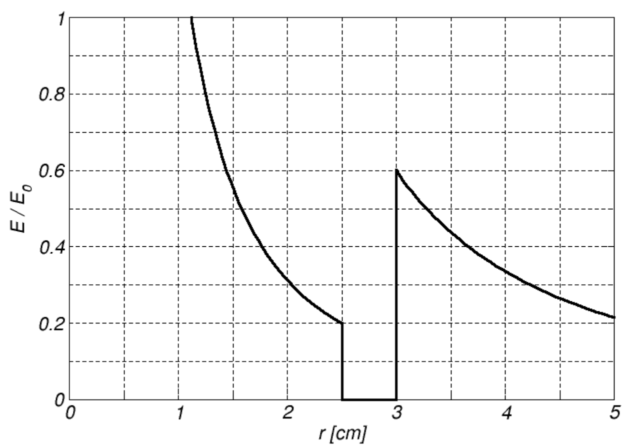
(b) whether the shell is metallic or dielectric;

(c) whether the shell is grounded or not;

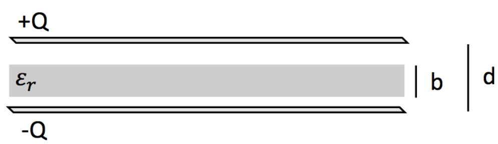
(d) the *total* charge on the shell;

(e) distribution of charge on the shell, make a sketch.

The vertical scale is in terms of $E_0 = 3 \cdot 10^4$ (SI units) and the Coulomb constant $k = 1/4\pi\varepsilon_0 = 9 \cdot 10^9$ (SI units).



(EM3) A parallel capacitor is made of two metal plates of area A separated by distance d. It is charged with $\pm Q$. A dielectric slab of thickness b ($b < d$) and dielectric constant $\epsilon_r (\equiv \epsilon/\epsilon_0) > 1$ is placed in between the plates, as shown in the figure.



(a) Sketch field lines of $\vec{E}$ (electric field), $\vec{D}$ (displacement field), and $\vec{P}$ (polarization) everywhere in the capacitor.

(b) Find the capacitance C of the capacitor – ignore edge effects. What is C at the limit $b = 0$ and $b = d$, respectively?

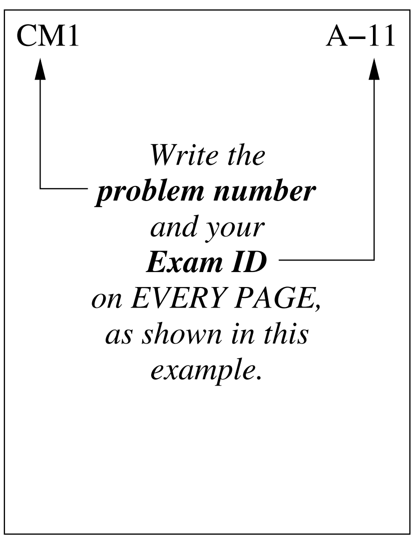
(c) Suppose that we first charge the capacitor to $\pm Q$ using a battery **without** the slab in place. We then remove the battery and slide the slab into place. Find the work done by the electric field to polarize the slab. Is this work positive or negative?

(d) If, on the other hand, the capacitor stays connected to the battery (of fixed voltage V) all the times, is the total electric field energy in the capacitor increased or decreased after the dielectric slab is placed in the capacitor? Explain why.

Department of Physics
Montana State University

Qualifying Exam
August, 2024

Day 4
Statistical and Thermal Physics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(ST1) Our goal is to find the equation of state of a white dwarf, which can be well approximated as a zero-temperature Fermi gas of non-interacting electrons. The equation of state is the relation between pressure P and the electron number density n_e, which follows a power-law relation $P \propto n_e^{\Gamma}$. Our goal is to find the value of Γ. Suppose the white dwarf has volume V and N electrons, so $n_e = N/V$. The electrons can be approximated as non-relativistic particles with its energy ϵ and momentum $p = |\boldsymbol{p}|$ related by $\epsilon = p^2/(2m_e)$.

(a). The Fermi energy ϵ_F is the maximum energy of an occupied electron state when the system is in its ground state. Compute ϵ_F in terms of $n_e = N/V$. [Hints: The spherical geometry of the star plays no role here, so you can use any geometry you like for the volume V – or none at all. If you are unable to do the exact calculation, you may try to estimate the answer based on the uncertainty principle. This will enable you to answer the rest of the problem.]

(b). Find E_t, the total energy of the system, and from it the averaged energy per particle E_t/N. How does E_t/N scale with the volume V?

(c). Find the power-law relation between pressure P vs density n_e, and the value of Γ. [Hint: You might find it convenient to find P in terms of E_t and V through the first law of thermodynamics.]

(d). Argue why we can focus only on electrons and ignore the proton/neutron contribution to the pressure.

(ST2) A system of N non-interacting magnetic dipole moments in external magnetic field has the energy

$$E^{mag} = -\sum_{i=1}^{N} \boldsymbol{\mu}_i \cdot \mathbf{B}\,,$$

where all magnetic moments have the same magnitude $|\boldsymbol{\mu}_i| = \mu$, but can point in arbitrary direction on the unit sphere. Treating magnetic moments as classical, take $\boldsymbol{\mu}_i \cdot \mathbf{B} = \mu B \cos\theta_i$ where $\theta_i \in [0, \pi]$ is the angle relative to the direction of the external field $\mathbf{B}$.

(a) Calculate the canonical partition function of a single magnetic moment in magnetic field $Z_1^{mag}(T, B)$, in equilibrium with thermostat T, and use it to find partition function of all N moments $Z_N^{mag}(T, B)$; (You can use limit $k_B T \gg \mu B$ to simplify your calculations and answers);

(b) Find the contribution from magnetic degrees of freedom to the entropy $S(T, B)$ in the limit $k_B T \gg \mu B$, leave only the first non-trivial term; assume the entropy of the system without magnetic degrees of freedom is known $S(T, 0) = S_0(T)$;

(c) Show that during an adiabatic process when magnetic field is reduced to zero, $B \to 0$, the magnetic material is cooled.

(ST3) An air conditioner works by circulating a working fluid, which we can approximate as an ideal gas, through a closed cycle of four steps: A–D. In step A the fluid is at the same temperature as the indoor air, $T_{\rm in}$, and *gains heat* from it; this *removes* heat from the air. In step B, the fluid is *adiabatically* compressed at entropy S_B up to the temperature of the outside air: $T_{\rm out} > T_{\rm in}$. The working fluid then exchanges heat with the outside air at $T_{\rm out}$ (step C), and finally is adiabatically returned (step D), at entropy S_D, to its original state at $T_{\rm in}$, and the same volume with which it began the cycle.

a. Draw a diagram in T *vs.* S space of the working fluid undergoing one complete cycle. Label each of the steps A–D described above on the diagram. State which step (or steps) require(s) a motor to **do positive work** on the working fluid.

b. In terms of $T_{\rm in}$, $T_{\rm out}$, S_B and S_D, compute the **heat** removed from the indoor air in step A.

c. Compute the **net work** done by the motor on the fluid over one complete cycle. Assume it works perfectly by recovering all *the work done on it by the fluid*.

d. Outside is a toasty $T_{\rm out} = 35°$, while indoors is kept at a pleasant $T_{\rm in} = 20°$. It is a perfect system (i.e. part c.) in which the motor draws an averaged power of 1000 W. At approximately what average rate is the air conditioner removing heat from the indoor air?