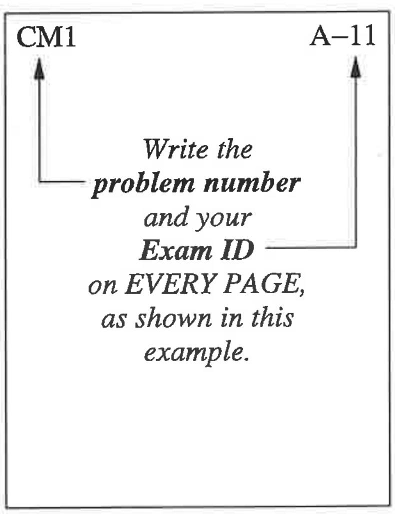
Department of Physics
Montana State University

Qualifying Exam
August, 2023

Day 1
Classical Mechanics



- Show your work.
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(CM1) A particle of mass m is confined to motion along the $+x$ axis, subject to a potential

$$V = \frac{a}{x} + \frac{b}{x^2},$$

where a and b are real constant.

(a) Graph the potential; consider how the shape of the potential varies for all combinations of signs of a and b. Note which choices of signs have equilibrium solutions.

(b) For each sign choice that admits an equilibrium, find all possible equilibrium positions.

(c) For one of the cases admitting a *stable* equilibrium, find the frequency of small oscillation around the equilibrium.

(CM2) A block of mass M can move left-right without friction. Identical springs, of spring constant k and rest length ℓ_0, connect the block to hard walls separated by distance $L > 2\ell_0$. An ideal simple pendulum, consisting of a massless rod of length length ℓ and a small bob of mass $m = M/2$, is suspended from the block — it swings in the plane of the diagram. Parameters are related as follows

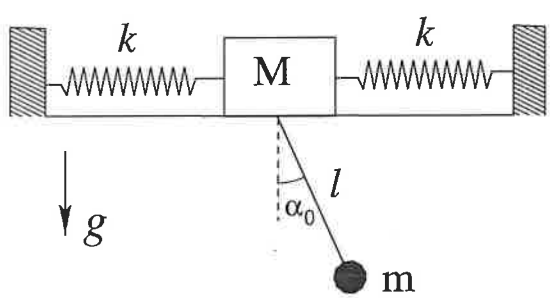
$$M = 2m \quad , \quad \frac{g}{\ell} = \frac{k}{m} = \omega_0^2$$

The system is initially positioned **at rest** in the configuration depicted in the figure. The block is at its midpoint and the pendulum is deflected from the vertical by angle $\alpha_0 \ll 1$. At $t = 0$ the block and pendulum are released from rest.

(a) Define a set of generalized coordinates. Use these to write the full potential and kinetic energies of the system, without assuming small angles.

(b) Assuming small perturbations, find the complete set of normal modes and eigenfrequencies of the system.

(c) Using the normal modes, find the position **of the block** for all times $t > 0$.



(CM3) A bowling ball of mass M and radius R is tossed onto a bowling lane so that it is initially sliding with velocity v_0 with no rotation. The lane has a coefficient of kinetic friction of μ and the ball is uniform, so the moment of inertia about its center is $I = (2/5)MR^2$.

(a) Find the equation describing the ball's time-dependent linear velocity as it is sliding.

(b) Find the equation describing the ball's angular velocity as it is sliding.

(c) Determine the distance the ball travels before it begins to roll without slipping and find its speed at that time.

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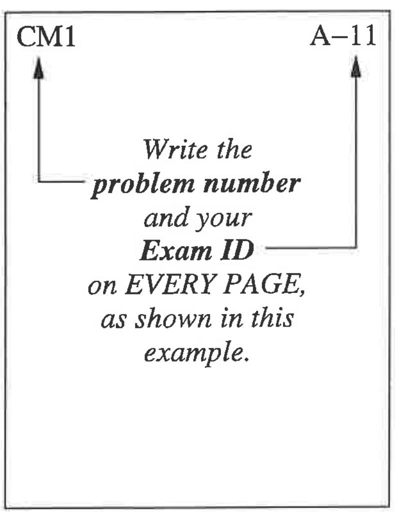
Day 2
Quantum Mechanics



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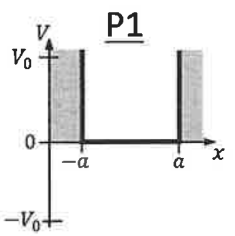
(QM1) Plotted on the following page are five 1D potentials labeled P1- P5 and seven spectra of bound states labeled S1-S7. The vertical and horizontal axes on all plots are the same. Ellipses on the spectra mean the states continue as $E \to \infty$

(a) For each potential, identify which spectrum describes the system. Briefly justify each assignment. You may only use a spectrum once.

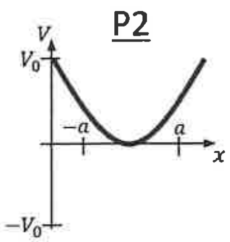
For the next two questions, label all axes and indicate the zeros for the x-axis and the y-axis. Within reason, be as descriptive as possible with your plots.

(b) For potential P5, sketch the wave function of the first excited state. Assume the potential is deep enough to have multiple bound states.

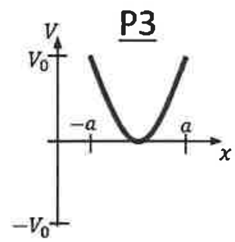
(c) For your answer in part (b), sketch the probability density for the position of the particle.



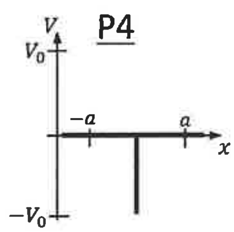
$$V(x) = \begin{cases} 0 & -a \le x \le a \\ \infty & otherwise \end{cases}$$



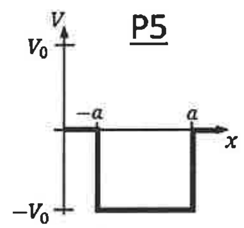
$$V(x) = \frac{1}{2}m(\omega_1)^2x^2$$



$$V(x) = \frac{1}{2}m(\omega_2)^2x^2$$



$$V(x) = -\alpha\delta(x - x_0)$$



$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & otherwise \end{cases}$$

(QM2) A photon can be in two polarization states $|v\rangle$ and $|h\rangle$, for vertical and horizontal. A photon with frequency ω enters a device that has vertical filters on both ends. Inside the device the photon polarization's behavior is governed by Hamiltonian (in the $|v\rangle\,, |h\rangle$-basis)

$$\mathcal{H} = \hbar\nu \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

where ν is a constant with dimension of 1/sec. It takes time τ for the photon to go from one end of the device to the other. What is the probability that the photon will come out from the other end of the device?

(QM3) A particle of mass m is contained in a one-dimensional, symmetric square well:

$$V_0(x) = \begin{cases} 0 & , \quad |x| < L \\ \infty & , \quad |x| > L \end{cases} \tag{1}$$

(a) Write down the complete set of energy levels, E_n, and **normalized** energy eigenstates, $\varphi_n(x)$ for the particle.

(b) The particle is subject to a an additional, small perturbing potential

$$V_1(x) = a\,\delta(x) \ , \tag{2}$$

where a is a constant and $\delta(x)$ is the Dirac-delta function. Use perturbation theory to write down the first order perturbations to the lowest three energy levels.

(c) Find a transcendental equation which is satisfied by the exact solutions E_n for the perturbed system (i.e. do not use perturbation theory). The equation can be for a variable *related to* E_n.

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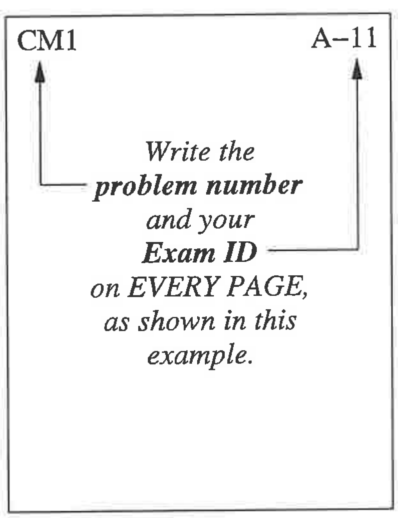
Day 3
Electricity and Magnetism



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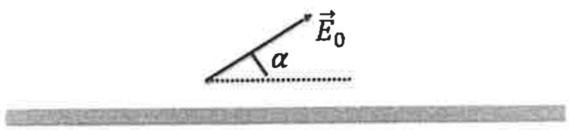
(EM1) A large thin dielectric slab of permittivity ϵ, area A, and thickness d ($d \ll \sqrt{A}$) is placed in an otherwise uniform electric field $\vec{E}_0$, and the angle between $\vec{E}_0$ and the slab plane is α, see the Figure.

(a) Find the electric field $\vec{E}_{in}$ inside the slab, ignoring edge effects.

(b) Find the bound charges.

(c) Find the **total** dipole moment of the entire slab. Compute the angle between the dipole moment and the normal to the slab's surface.



(EM2) We have a thin wire loop given by $x^2 + y^2 = a^2$. A small permanent magnet of dipole moment m is placed at $(s, 0, 0)$ with $s \gg a$. The loop has resistance R and negligible self-induction. Initially there is no current in the loop, and the magnet is pointing along the z-axis. The magnet is then slowly (quasi-statically) rotated to point in the opposite direction:

$$m\hat{z} \quad \rightarrow \quad -m\hat{z}.$$

(a) Find the total charge that passes through a certain cross-section of the loop wire in this process.

(b) Show that under the assumptions given above the answer does not depend on the exact way the magnet is rotated, but only on its initial and final orientations.

(EM3) An electromagnetic plane wave, incident on a planar interface, has wave vector

$$\mathbf{k}_\mathrm{i} \; = \; k_0\hat{\mathbf{x}} \; + \; 2k_0\hat{\mathbf{z}} \;\; , \tag{3}$$

where $k_0 > 0$. The wave is incident from a half-space of vacuum, $z < 0$, while the other half-space, $z > 0$, is filled with a linear dielectric, $\epsilon = 2\epsilon_0$. (Both sides have permeability μ_0). The wave is polarized in the plane of incidence, and at the origin the electric field of the incident wave is

$$\mathbf{E}_i(\mathbf{0}, t) = E_0 \left(2\hat{\mathbf{x}} - \hat{\mathbf{z}}\right) \cos(\omega t) \;\; , \tag{4}$$

where $E_0 > 0$ and $\omega = c|\mathbf{k}_i|$.

(a) Write down **full** expressions for the **incident** electric **and** magnetic field for all points in the vacuum and all times. Express the amplitude of the magnetic field in terms of E_0 and other constants given in the problem.

(b) Write down **full** expressions for both the **transmitted and reflected** electric fields in their respective spatial domains. Use the jump conditions across the interface to express all component amplitudes in terms of only E_0.

(c) Write down the surface charge density, if any, within the $z = 0$ surface.

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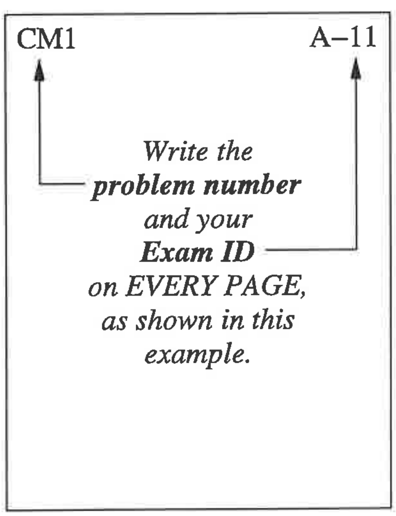
Day 4
Statistical and Thermal Physics



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(ST1) The photon gas is used as a working medium in Carnot engine. The equation of state for the gas is

$$PV = \frac{1}{3}E \,, \qquad E = aVT^4$$

where P is pressure, V - volume, E - internal energy T - temperature and a - is a parameter made up from the universal constants.

(a) Plot the Carnot cycle in ST (entropy-temperature) coordinates and indicate on the plot the part that corresponds to the adiabatic compression of the gas.

(b) If the initial volume was V_i find the final volume of the gas after this compression; find the work done *on* the gas in this process.

(c) Use the first law of thermodynamics and the plot in part (a) to calculate work done *by* the engine in one full cycle, and find the engine's efficiency. Is it different from efficiency of Carnot engine that has usual gas as working medium? How?

Recall:

Carnot engine takes heat from reservoir at T_2 and releases heat into reservoir at T_1. Between heat exchange processes are adiabatic compression and expansion.

Efficiency is the ratio of work done by the engine and the heat supplied to the engine, during one cycle.

(ST2) Consider a system of a large number $N = n_1 + n_2$ of two-level 'particles'. n_1 particles are in the state with energy E_1 and n_2 particles are in the state with energy E_2. The system is connected to a heat reservoir that is held at temperature T. The system of N particles undergoes a change where one particle transitions from the state with energy E_1 to the state with energy E_2:

$$\begin{aligned} n_1 &\rightarrow n_1 - 1 \\ n_2 &\rightarrow n_2 + 1 \end{aligned}$$

(a) For this process calculate the change in entropy to the system of N particles in terms of N, n_1, and n_2.

(b) Calculate the change in entropy to the heat reservoir due to the heat transferred from the particles to the heat reservoir.

(c) From (a) and (b), **derive** the Boltzmann relation for the ratio n_1/n_2. (i.e. derive it using answers to (a) and (b) without assuming it in advance.)

Small hint: For large n, $n \pm 1 \approx n$, and you might or might not need the Stirling's approximation $\ln n! \approx n \ln n - n$.

(ST3) A system of N identical spin-half particles have magnetic dipole moment and can therefore exist in states with dipole moment $+m_0$ and $-m_0$. In a magnetic field H these states have energy $-m_0 H$ and $+m_0 H$ respectively. These are the only two energy states of a given particle.

(a) Write down the partition function for this system.

(b) Use the partition function from (a) to find an expression for the energy of the system.

(c) The N particles are distributed uniformly through a volume V. Use the probability of each magnetic moment to find the **magnetization** M — the **average** magnetic moment per unit volume.

(d) Using the results of (c) show that for sufficiently small magnetic field strength H, the magnetization is proportional to field strength

$$M \simeq \chi_m H \ , \tag{5}$$

where the magnetic susceptibility χ_m depends on temperature.