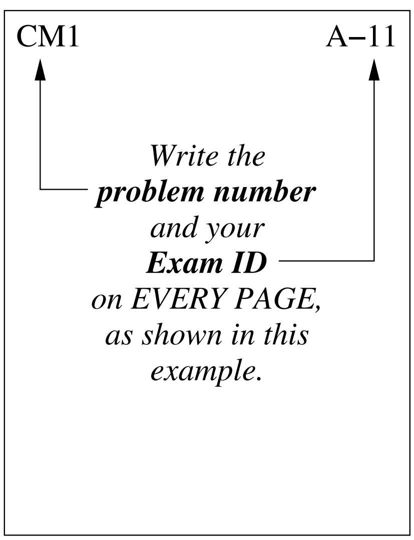
Department of Physics
Montana State University

Qualifying Exam
January, 2025

Day 1
Classical Mechanics



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(CM1) A periodically driven, damped harmonic oscillator of mass m and spring constant k satisfies the equation

$$m\,\ddot{x} \;=\; -k\,x \;-\; m\,\nu\,\dot{x} \;+\; F_0\,\cos(\omega t) \;\;,$$

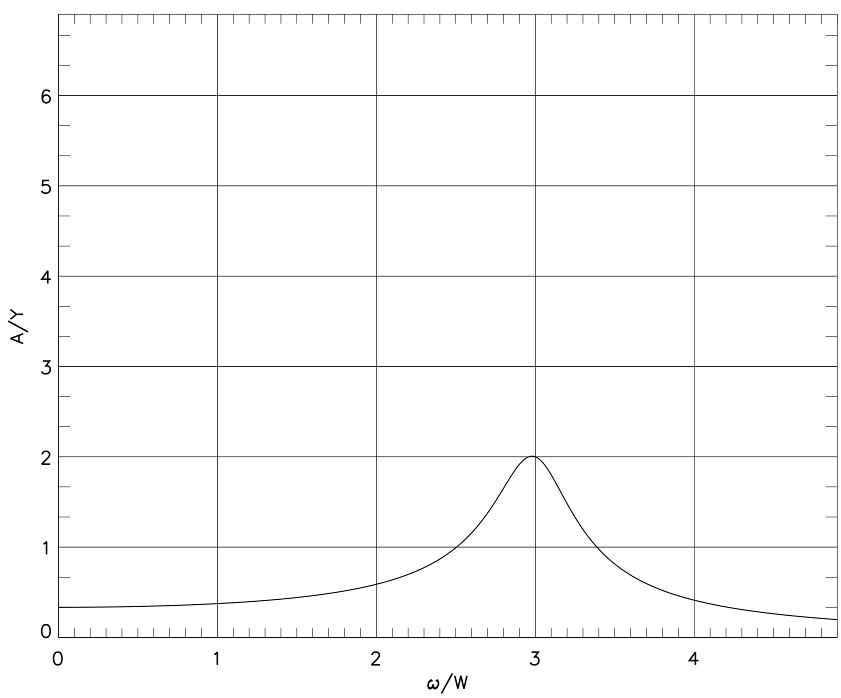
where ν is a damping coefficient, and F_0 and ω are the amplitude and frequency of driving. The solution can be written in the form

$$x(t) \;=\; A\,\cos\Big[\,\omega t \;-\; \phi\,\Big] \;\;,$$

where A and ϕ are both real and depend on the parameters of the problem.

a. Find an expression for $A(\omega)$ in terms of the parameters of the problem.

b. The figure below is a plot of $A(\omega)$ for values $m = 0.1\,\text{kg}$, $k = 0.9\,\text{N/m}$, $\nu = 0.5\,\text{s}^{-1}$, and $F_0 = 0.45$ N. Axes are scaled to Y and W. Use the results of a. to write the values of Y and W in SI units (i.e. m, kg, s).

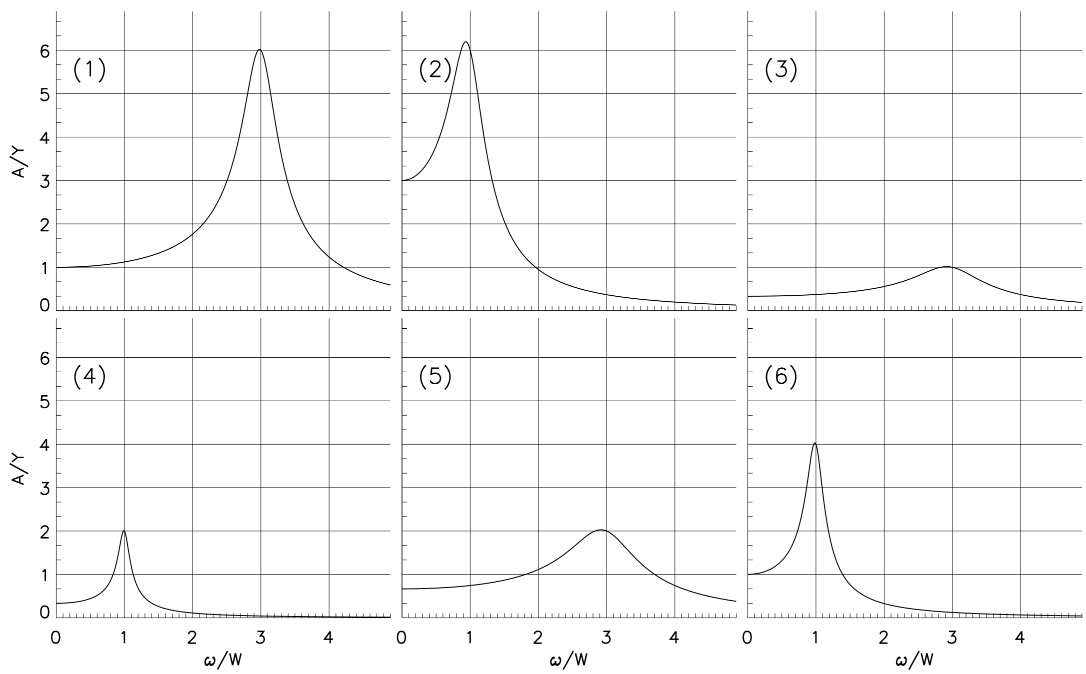


(Continued on next page)

The figure below shows versions of $A(\omega)$ using the same values of Y and W found in part b., but in each case one parameter has been changed.
Please match the appropriate figure to the cases described below. Write text justifying your choice.

c. All parameters are the same as in b., except $k = 0.1\,\mathrm{N/m}$.

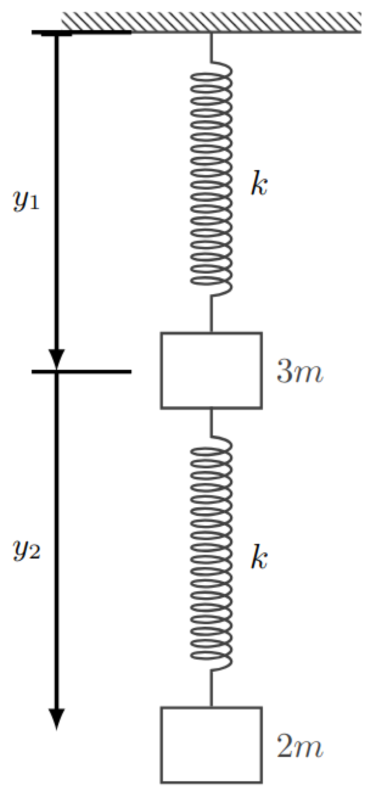
d. All parameters are the same as in b., except $\nu = 1.0\,\mathrm{s}^{-1}$.



(CM2) A particle of mass $3m$ is suspended from a fixed point O by a light linear spring with spring constant k. A second particle of mass $2m$ is in turn suspended from the first particle (the one of mass $3m$) by a second spring that is identical to the first spring. The system only moves in the vertical direction and is subject to gravity.

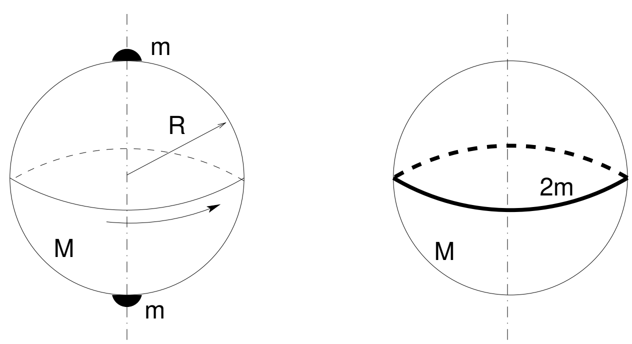
a. Let y_1 denote the distance of the first particle from the mounting point, y_2 represent the distance of the second particle from the first particle (as shown in the sketch below) and l represent the relaxed spring length. Using these coordinates, demonstrate that the Lagrangian is:

$$L = \frac{3m}{2}\dot{y}_1^2 + m(\dot{y}_1 + \dot{y}_2)^2 + 3mgy_1 + 2mg(y_1 + y_2) - \frac{1}{2}k(y_1 - l)^2 - \frac{1}{2}k(y_2 - l)^2$$



b. Find the equilibrium position of the two masses from the equation of motion.

(CM3) Consider a free sphere spinning about vertical axis with period T. The sphere is solid, with mass M and radius R, and moment of inertia $I = (2/5)MR^2$. It has two point-like masses m initially sitting at each pole. Gradually these masses "flow" from poles to the equatorial region to form a single thin uniform ring around the entire equator, spinning *together* with the sphere.
Find the period of the rotation in the new configuration. How do the rotational energy and the angular momentum of the system change between initial and final states? If they change, suggest a mechanism how this happens.



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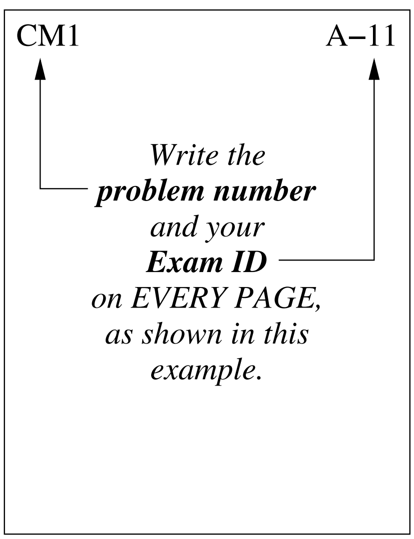
Day 2
Quantum Mechanics



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(QM1) A spin 1/2 particle, with gyromagnetic ratio γ, is subject to a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ giving it a Hamiltonian

$$\hat{H} \;=\; -\gamma B_0\,\hat{S}_z \quad . \tag{1}$$

S_y is measured at $t = 0$ and found to be $+\hbar/2$.

a. Write the state $|\psi(t)\rangle$ for $t \geq 0$ in terms of normalized eigenstates of $\hat{H}$.

b. Use the result of a. to compute $\langle S_x \rangle$ as a function of time for $t \geq 0$.

The spin operators, expressed in $|\uparrow\rangle, |\downarrow\rangle$ basis, are given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(QM2) An electron is confined to a 2D infinite square well with infinite potential barriers at $x = -a/2, a/2$ and $y = -a/2, a/2$. A weak perturbation exists in the system of the following form:

$$\hat{H}' = V_0 \hat{x}\hat{y}$$

where V_0 is a positive, real constant.

Determine how the perturbation changes the energies and wave functions of the first two excited states to lowest order in V_0.

Note: you may find some of the below integrals useful:

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x\right) x \sin\left(\frac{2\pi}{a}x\right) dx = \frac{8a^2}{9\pi^2}$$

$$\int_{-a/2}^{a/2} \cos\left(\frac{2\pi}{a}x\right) x \sin\left(\frac{\pi}{a}x\right) dx = -\frac{10a^2}{9\pi^2}$$

$$\int_{-a/2}^{a/2} \sin\left(\frac{2\pi}{a}x\right) x \sin\left(\frac{2\pi}{a}x\right) dx = -\frac{a^2}{8\pi}$$

(QM3) A quantum harmonic oscillator is prepared in a state

$$|\nu\rangle = e^{-|\nu|^2/2} \sum_{n=0}^{\infty} \frac{\nu^n}{\sqrt{n!}} |n\rangle \qquad \text{with} \qquad \nu = \sqrt{N} e^{i\theta} \,.$$

Here N and θ are arbitrary real numbers, and $|n\rangle$ are the orthonormal eigenstates of the oscillator Hamiltonian $\mathcal{H} = \hbar\omega(a^\dagger a + 1/2)$, with $a^\dagger a|n\rangle = n|n\rangle$, where $a^\dagger, a$ are raising and lowering operators, with commutation relation $\left[a, a^\dagger\right] = 1$.

(a) In terms of N, θ, what is the probability of finding the oscillator in eigenstate n?

(b) Show that $|\nu\rangle$ is properly normalized. Recall Taylor expansion of the exponential function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

In the remaining questions use the fact that $|\nu\rangle$ is an eigenstate of operator a (you don't have to prove it):

$$a|\nu\rangle = \nu|\nu\rangle \qquad \text{and its adjoint} \qquad \langle\nu|a^\dagger = \langle\nu|\nu^* \,.$$

Note that a acts to the right and $a^\dagger$ acts to the left to get the eigenvalue property.

(c) Find the expectation value $\bar{E}$ of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$.

(d) Find the variance σ_E^2 of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$. (It'll help to use the commutation relation to write $a^\dagger a a^\dagger a = a^\dagger a^\dagger a a + a^\dagger a$ so that all a can freely act to the right on $|\nu\rangle$, and $a^\dagger$ can act to the left on $\langle\nu|$).

(e) Based on (c) and (d), is $|\nu\rangle$ an energy eigenstate of the oscillator? Does it agree with (a)? Using $\sigma_E/\bar{E}$ ratio argue whether $|\nu\rangle$ looks like an eigenstate of the oscillator in the limit of small or large N?

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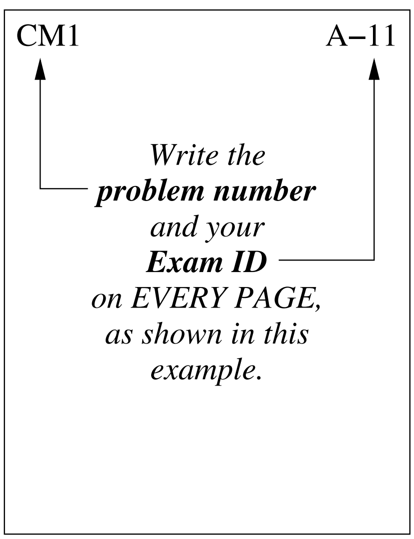
Day 3
Electricity and Magnetism



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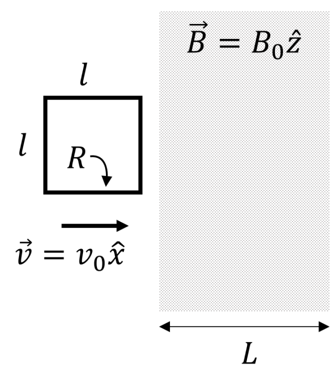
(EM1) A square loop with resistance R and side length l is pulled slowly toward a region of uniform magnetic field with a **constant** speed v_0. The magnetic field has a strength B_0 and is oriented orthogonal to the plane of the loop. The width of the region with non-zero magnetic field is L, where $L = 2l$. Assume that at time $t = 0$, the right edge of the loop is just about to enter the magnetic field. Also assume that the self-inductance of the loop is negligible.



(a) Determine the magnitude and direction of the electric current I in the loop for all times. Plot the magnitude of the current as a function of time.

(b) Determine the total energy dissipated by the loop as it traverses through the region with magnetic field.

(EM2) A capacitor is made of a pair of concentric spherical shells of radius a and b ($a < b$). Both shells are perfect conductors, and a dielectric material of dielectric constant $\epsilon_r > 1$ fills the space between the two shells. The capacitor carries charge $\pm Q$ on the inner and outer shells, respectively.

(a) Find the displacement field $\vec{D}$ **and** the electric field $\vec{E}$, and graph the magnitude of the electric field as a function of r, the distance to the center of the sphere.

(b) Find the potential difference, V, between the two shells.

(c) Find the capacitance of the capacitor.

(d) At the limit $\epsilon_r \to \infty$, what will your solutions mean?

(EM3) A transmission line is made of a pair of long coaxial cylindrical shells with the inner radius a and outer radius b. Both shells are conductors. The electric field between the two conducting shells is found to be

$$\vec{E} = E_0(s)\cos(kz - \omega t)\hat{s},$$

where s is the distance to the axis of the cables.

(a) From Gauss's law, show that $E_0(s) = A_0/s$, where A_0 is a constant.

(b) From Faraday's law, find the magnetic field $\vec{B}$.

(c) Sketch the electric field lines and magnetic field lines at several locations along the cable, at $t = 0$.

(d) Find the Poynting vector, its magnitude and direction.

Note: the divergence and curl of a vector in cylindrical coordinates are given by:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A} &= \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \\
\vec{\nabla} \times \vec{A} &= \hat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right) + \hat{z}\left[\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) - \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right].
\end{aligned}$$

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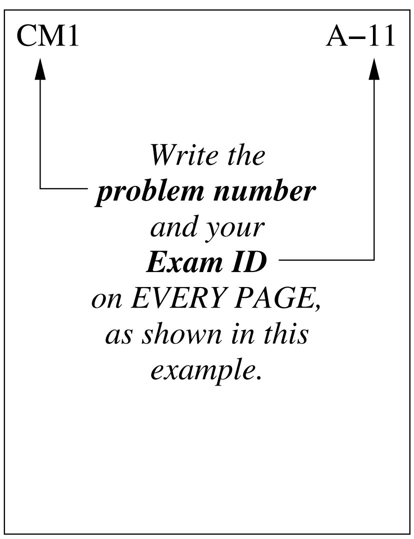
Day 4
Statistical and Thermal Physics



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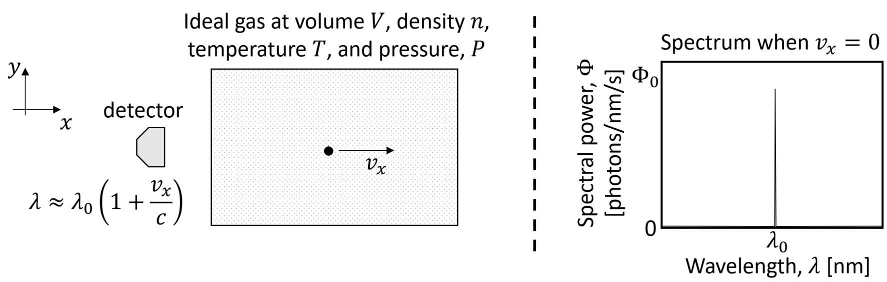
(ST1) Consider an ideal gas with molar density n, volume V, pressure P, and temperature T. The gas is composed of atoms that in the rest frame emit light at a wavelength λ_0. The wavelength of the emitted light from a moving atom, λ, will be Doppler-shifted depending on its relative velocity with respect to a detector as shown below.



Determine the full-width-at-half-max (FWHM) of the spectrum of light from the ideal gas that is observed by the detector, and plot how it depends on temperature. Consider only the effects of velocity in the x direction.

Note 1: The emission spectrum of the detected light is given as the *spectral power*, $\Phi(\lambda)$, where the number of photons (per unit time), dN_ν, emitted at a wavelength λ over the range $d\lambda$ is:

$$dN_\nu = \Phi(\lambda) \cdot d\lambda$$

Note 2: the FWHM of a peak is determined by calculating the separation between the two points of the peak that have 1/2 of the max value.

(ST2) The goal of this problem is to explain why prior to development of quantum mechanics the heat capacity of diatomic gases was a puzzle. Let's assume we are dealing with nitrogen molecule N_2. The relevant degree of freedom is stretching motion along the molecule's axis; neglect all other molecular motions.

(a) Using the equipartition theorem, write the classical heat capacity of the 1-dimensional oscillators per 1 mole.

(b) Quantum mechanically, the spectrum E_n ($n = 0, 1, 2 \dots$) of an oscillator is determined by oscillation frequency ω. The oscillator is in thermal equilibrium with thermostat at temperature T. What is the probability to find the oscillator in state n?

(c) Find and sketch the expectation value of the N_2 oscillator's energy E vs temperature. For nitrogen $\hbar\omega/k_B \sim 3500\,K$.

(d) Using the sketch of $E(T)$, graphically determine the specific heat of N_2 and explain what was missing in the classical picture at room temperature $\sim 300\,K$?

(ST3) An air condition works by circulating a working fluid, which we can approximate as an ideal gas, through a closed cycle of four steps: A–D. The fluid begins the cycle at the same temperature as the **indoor** air, T_{in}. In step A it is *adiabatically* compressed at entropy S_A up to the temperature of the outside air: $T_{\text{out}} > T_{\text{in}}$. In step B the working fluid exchanges heat with the outside air at T_{out}. In step C it is adiabatically cooled at entropy S_C going from $T_{\text{out}} \to T_{\text{in}}$. Finally, in step D the fluid at the same temperature as the **indoor** air **gains heat** from it; this *removes* heat from the indoor air.

a. Draw a diagram in T *vs.* S space of the working fluid undergoing one complete cycle. Label each of the steps A–D described above on the diagram. Which step (or steps) require(s) a motor to **do positive work** on the working fluid?

b. In terms of T_{in}, T_{out}, S_A and S_C, compute the heat removed from the indoor air in step D.

c. Compute the **net work** done by the motor on the fluid over one complete cycle. Assume it works perfectly by recovering all *the work done on it by the fluid.*

d. Outside is $T_{\text{out}} = 30°$ C, while indoors is kept at kept at a pleasant $T_{\text{in}} = 20°$ C. In a perfect system (i.e. part c.) the motor draws 500 W of power. At approximately what rate is the air conditioner removing heat from the indoor air?