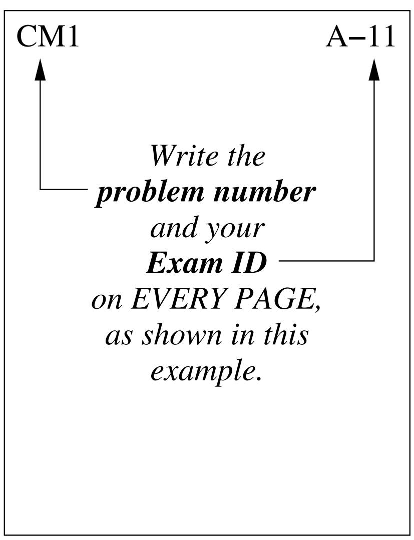
Department of Physics
Montana State University

Qualifying Exam
August, 2024

Day 1
Classical Mechanics



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(CM1) A harmonic oscillator has natural frequency ω_0 and a weak damping $0 < \gamma \ll \omega_0$. It is driven by a force at a single frequency, with $F(t) = \omega_0^2 f_0 e^{i\Omega t}$, with f_0 a constant. The equation of motion reads

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \omega_0^2 f_0 e^{i\Omega t}.$$

[Here we work with a complex x for convenience. All the frequencies $(\omega_0, \Omega, \gamma)$ are real. To get physical solutions, we can simply take the real part of x. You may or may not need Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$.]

(a). When $f_0 = 0$ (i.e., without external drive), find a general expression for $x(t)$ for arbitrary, non-trivial initial conditions.

(b). For non-zero f_0, show that a solution of the form $x(t) = B\exp(i\Omega t)$ solves the equation. Express the complex amplitude B in terms of other parameters of the problem.

(c). Show that the sum of the solutions from (a) and (b) also solves the equation. Suppose we wait long enough, which term in the sum dominates?

(d). Consider only the dominant term from (c). Define the response function $T(\Omega) = x/f_0$. Sketch out the shape of $|T(\Omega)|$ as a function of $\Omega > 0$. When will $|T(\Omega)|$ achieve its maximum and what is its value there? Make sure you also include the correct asymptotic behaviors in the limit $\Omega \ll \omega_0$ and $\Omega \gg \omega_0$. In the two limits, is $x(t)$ in phase or out of phase with $F(t)$?

Solution:

(a). The solution has the form $x = Ae^{i(\omega t + \phi_0)}$ with $\omega \in \mathbb{C}$.

$$-\omega^2 + 2i\gamma\omega + \omega_0^2 = 0, \tag{1}$$

so

$$\omega = \pm\sqrt{\omega_0^2 - \gamma^2} + i\gamma \simeq \omega_0 + i\gamma. \tag{2}$$

Therefore, the general solution reads

$$x(t) = Ae^{i\phi_0}e^{-\gamma t}e^{\pm i\sqrt{\omega_0^2-\gamma^2}}, \tag{3}$$

or equivalently,

$$\mathrm{Re}\,[x(t)] = e^{-\gamma t}\left[A_c\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) + A_s\sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right)\right], \tag{4}$$

where (A, ϕ_0) or (A_c, A_s) are arbitrary constants to be set by the initial conditions.

(b). From the provided ansatz,

$$\left(-\Omega^2 + 2i\gamma\Omega + \omega_0^2\right) B = \omega_0^2 f_0, \tag{5}$$

so

$$B = \frac{\omega_0^2 f_0}{\omega_0^2 - \Omega^2 + 2i\gamma\Omega}. \tag{6}$$

(c). Suppose the solutions in (a) and (b) are denoted respectively by x_a and x_b, satisfying

$$\ddot{x}_a + 2\gamma\dot{x}_a + \omega_0^2 x = 0. \tag{7}$$

$$\ddot{x}_b + 2\gamma\dot{x}_b + \omega_0^2 x_b = \omega_0^2 f_0 e^{i\Omega t}. \tag{8}$$

then summing the two equations leads to

$$(\ddot{x}_a + \ddot{x}_b) + 2\gamma(\dot{x}_a + \dot{x}_b) + \omega_0^2(x_a + x_b) = \omega_0^2 f_0 e^{i\Omega t}. \tag{9}$$

Since taking time derivatives is a linear operation, $x_a + x_b$ is also a solution of the equation of motion.

As $t \to +\infty$, $x_a \propto e^{-\gamma t}$ will decay away, and therefore x_b dominates the solution.

(d).

$$T(\Omega) = \frac{B}{f_0} = \frac{\omega_0^2}{\omega_0^2 - \Omega^2 + 2i\gamma\Omega} = \frac{\omega_0^2(\omega_0^2 - \Omega^2 - 2i\gamma\Omega)}{(\omega_0^2 - \Omega^2)^2 - 4\gamma^2\Omega^2}. \tag{10}$$

A sketch of $|T(\Omega)|$ is shown in Fig. 1.

When $\Omega \ll \omega_0$, $T \simeq 1$, and the oscillator moves in phase with the drive.

When $\Omega \gg \omega_0$, $T \simeq -\omega_0^2/\Omega^2$. In other words, a high-frequency drive is suppressed as Ω^{-2}. The oscillator moves in the opposite phase with the drive.

When $\Omega \simeq \omega_0$, the value of $|T|$ is maximized. This corresponds to resonance. The peak value is

$$|T_{\max}| = |T(\Omega = \omega_0)| = \frac{\omega_0}{2\gamma}. \tag{11}$$

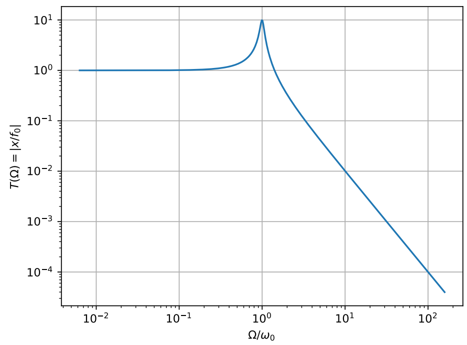


Figure 1: Response $T(\Omega)$.

It is not required but we can also identify further the half width at half maximum $\Delta\omega$ of the resonance,

$$|(\omega_0 \pm \Delta\Omega)^2 - \omega_0^2| \simeq 2\gamma\omega_0, \text{ or } \Delta\Omega \simeq \gamma. \tag{12}$$

(CM2) As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Figure 2, with two identical atoms each of mass m connected by two identical springs (with spring constant K and rest length l_0) to a single atom of mass M. Assume that all motion is in the x direction and use the positions of the masses x_1, x_2 and x_3, as generalized coordinates.

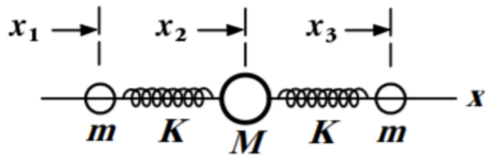


Figure 2: Cartoon of a CO2 molecule.

(a). Write down the Lagrangian and find the equations of motion.

(b). Find the normal frequencies of the system. Assuming $m \simeq M$, rank the magnitude of the eigenfrequencies from the highest to the lowest. Show that one of the normal frequencies is zero.

(c). Find the normal mode vectors. (You need not normalize them.) Describe in words the motion corresponding to each nomral mode.

Solution:

Kinetic energy T is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2. \tag{1}$$

Potential energy U is

$$U = \frac{1}{2}k\left(x_1 - x_2\right)^2 + \frac{1}{2}k(x_2 - x_3)^2. \tag{2}$$

The Lagrangian is $L = T - U$.

Equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1 = \frac{\partial L}{\partial x_1} = -k(x_1 - x_2), \tag{3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = M\ddot{x}_2 = \frac{\partial L}{\partial x_2} = k(x_1 - x_2) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3), \tag{4}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_3} = m\ddot{x}_3 = \frac{\partial L}{\partial x_3} = k(x_2 - x_3), \tag{5}$$

or in a matrix form,

$$\underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}}_{\boldsymbol{M}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -k \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{\boldsymbol{K}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix}. \tag{6}$$

(b). Assuming $x \sim e^{i\omega t}$, the eigenfrequencies are found through the roots of

$$\det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) = 0, \tag{7}$$

or

$$\omega^2 (\frac{k}{m} - \omega^2)(\frac{2m + M}{mM} k - \omega^2) = 0. \tag{8}$$

Therefore, in descending order the eigenfrequencies are

$$\omega_1^2 = \frac{2m + M}{mM} k, \omega_2^2 = \frac{k}{m}, \omega_3^2 = 0. \tag{9}$$

(c). Find eigenmodes ξ_i through

$$\left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M}\right) \boldsymbol{\xi}_i = 0. \tag{10}$$

We have,

$$\left(\boldsymbol{K} - \omega_1^2 \boldsymbol{M}\right) = \begin{pmatrix} -2\frac{m}{M}k, & -k, & 0 \\ -k & -\frac{M}{m}k, & -k \\ 0, & -k, & -2\frac{m}{M}k \end{pmatrix}, \tag{11}$$

so the unnormalized eigenmode

$$\boldsymbol{\xi}_1 = (1, -2\frac{m}{M}, 1)^T. \tag{12}$$

In other words, this mode corresponds to the two atoms on both sides (i.e., the two m's) moving in the same direction, while the central atom M moving in the opposite direction.

For the next one,

$$\left(\boldsymbol{K} - \omega_2^2 \boldsymbol{M}\right) = \begin{pmatrix} 0, & -k, & 0 \\ -k & -(2 - \frac{M}{m})k, & -k \\ 0, & -k, & 0 \end{pmatrix}, \tag{13}$$

so

$$\boldsymbol{\xi}_2 = (1, 0, -1)^T. \tag{14}$$

In this case, M stays at rest while the two m's moving in opposite directions with the same amplitude.

Lastly,

$$\left(\boldsymbol{K} - \omega_3^2 \boldsymbol{M}\right) = \begin{pmatrix} k, & -k, & 0 \\ -k & 2k, & -k \\ 0, & -k, & k \end{pmatrix}, \tag{15}$$

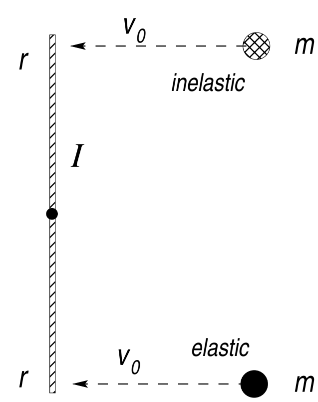
so

$$\boldsymbol{\xi}_3 = (1, 1, 1)^T. \tag{16}$$

This means the entire molecule moves as a whole without internal motion.

(CM3) Two balls hit a rod that can freely pivot around its center. Both balls have the same mass m and speed v_0. They hit perpendicular to the rod at equal distances r from the pivot axis. One ball undergoes a perfectly elastic collision with the rod and the other a perfectly inelastic (i.e. it sticks to the rod). Assume each collision happens instantaneously, but the elastic collision happens just before the inelastic. The time between the collisions is very short, so that the rod does not change its position significantly during that interval, and both balls hit the rod at 90 degree angles.

What will be the eventual direction and angular speed of rotation of the rod and stuck ball? The moment of inertia of the rod about its pivot is $I \gg mr^2$.



Solution:

Let's choose the coordinate system. Take positive direction of linear velocity to be from right to left, and positive angular velocity corresponding to anti-clockwise rotation.

An elastic collision is the one where kinetic energies of colliding bodies are conserved, while inelastic collision will lose kinetic energy and the two colliding objects will move together after the collision. In both cases the angular momentum is conserved.

We can make the following table for the velocities of the three objects before and after each collision. We treat both collisions as instantaneous,

	rod	top ball (inelastic)	bottom ball (elastic)
Before collisions	$\omega_0 = 0$	v_0	v_0
After elastic collision	$\tilde{\omega} =?$	v_0	$v =?$
After inelastic collision	$\omega =?$	$\omega r =?$	n/a

For the first, elastic collision, we write two conservation laws:

$$-mrv_0 = I\tilde{\omega} - mrv \qquad \text{(angular momentum)}$$
$$\frac{1}{2}mv_0^2 = \frac{1}{2}I\tilde{\omega}^2 + \frac{1}{2}mv^2 \qquad \text{(energy)}$$

and solve for intermediate angular velocity $\tilde{\omega}$ and the outgoing velocity of the bottom ball:

$$\tilde{\omega} = -\frac{2mrv_0}{I + mr^2}$$
$$v = v_0\frac{mr^2 - I}{mr^2 + I}$$

Negative angular velocity means rod rotates clockwise.

During the inelastic collision only angular momentum is conserved and the ball 'sticks' to the rod, moving with the same angular velocity after the collision

$$I\tilde{\omega} + mv_0r = I\omega + mr^2\omega$$

This gives final answer

$$\omega = \frac{I\tilde{\omega} + mv_0r}{I + mr^2} = -\frac{I - mr^2}{(I + mr^2)^2}mrv_0$$

For the $I \gg mr^2$ the angular velocity will be $\omega \approx -mrv_0/I$ - clockwise rotation, since it is negative.

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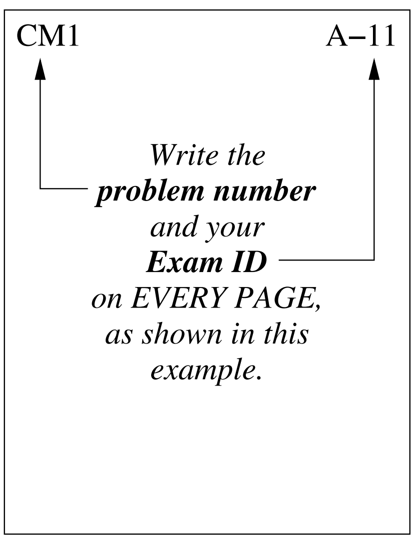
Day 2
Quantum Mechanics



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(QM1) A particle of mass m is in a 2d $a \times a$ square well potential with infinite walls at $x = 0$, $x = a$, $y = 0$, $y = a$. The particle is subject to a perturbation

$$V'(x,y) \ = \ \alpha\, \delta(x-y) \quad , \tag{1}$$

where $\alpha > 0$ is real. Find **all** the states in the **first excited energy level** of the **unperturbed** system. Use degenerate perturbation theory to compute the first order corrections to the energies of the first excited state.

Some or all of the following integrals may prove helpful $(m, n = 1, 2, 3, \cdots)$

$$\int_0^1 \sin^2(n\pi t)\, dt \ = \ \int_0^1 \cos^2(n\pi t)\, dt \ = \ \frac{1}{2}$$

$$\int_0^1 \sin^2(n\pi t)\, \sin^2(m\pi t)\, dt = \frac{1}{4} + \frac{1}{8}\delta_{nm}$$

Solution:

The first excited state must have one dimension with $n = 1$ and the other with $n = 2$. The possibilities are

$$\psi_1(x,y) \ = \ \frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right) \tag{1}$$

$$\psi_2(x,y) \ = \ \frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{2}$$

We must work out all matrix elements of the perturbation. One diagonal elements is

$$V_{11} \;=\; \langle\psi_1|\,V'\,|\psi_1\rangle \;=\; \frac{4\alpha}{a^2}\int\!\!\int \sin^2\left(\frac{\pi x}{a}\right)\,\sin^2\left(\frac{2\pi y}{a}\right)\,\delta(x-y)\,dx\,dy$$

$$\;=\; \frac{4\alpha}{a^2}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\,\sin^2\left(\frac{2\pi x}{a}\right)\,dx \;=\; \frac{\alpha}{a^2}\int_0^a \left[\cos\left(\frac{\pi x}{a}\right) \;-\; \cos\left(\frac{3\pi x}{a}\right)\right]^2 dx$$

$$\;=\; \frac{\alpha}{a^2}\int_0^a \left[\cos^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{3\pi x}{a}\right) - 2\,\cos\left(\frac{\pi x}{a}\right)\,\cos\left(\frac{3\pi x}{a}\right)\right] dx$$

$$\;=\; \frac{\alpha}{a} \;-\; \frac{\alpha}{a^2}\int_0^a \left[\cos\left(\frac{2\pi x}{a}\right) + \cos\left(\frac{4\pi x}{a}\right)\right] dx \;=\; \frac{\alpha}{a} \qquad (3)$$

It is easy to see that $V_{22} = V_{11}$. One diagonal element is

$$V_{12} \;=\; \langle\psi_1|\,V'\,|\psi_2\rangle \;=\; \frac{4\alpha}{a^2}\int\!\!\int \sin\left(\frac{\pi x}{a}\right)\,\sin\left(\frac{2\pi y}{a}\right)\,\sin\left(\frac{2\pi x}{a}\right)\,\sin\left(\frac{\pi y}{a}\right)\,\delta(x-y)\,dx\,dy$$

$$\;=\; \frac{4\alpha}{a^2}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\,\sin^2\left(\frac{2\pi x}{a}\right)\,dx \;=\; V_{11} \;=\; \frac{\alpha}{a} \;. \qquad (4)$$

Finally,

$$V_{21} \;=\; \langle\psi_2|\,V'\,|\psi_1\rangle \;=\; \langle\psi_1|\,V'\,|\psi_2\rangle^* \;=\; \left(\frac{\alpha}{a}\right)^* \;=\; \frac{\alpha}{a} \qquad (5)$$

The energy perturbations are the eigenvalues of this matrix

$$\det(\underline{\underline{V}} - E\,\underline{\underline{I}}) \;=\; \det\begin{bmatrix} (\alpha/a) - E & \alpha/a \\ \alpha/a & (\alpha/a) - E \end{bmatrix} \;=\; E\,(E - 2\alpha/a) \;=\; 0 \;. \qquad (6)$$

The perturbed energies are, therefore

$$E_a^1 \;=\; 0 \quad , \qquad E_b^1 \;=\; \frac{2\alpha}{a} \;. \qquad (7)$$

(QM2) A particle of mass m is bound in a harmonic 1D potential of the form:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

At time $t = 0$, the particle is in the following state:

$$|\Psi(0)\rangle = A\Big(|0\rangle + 2\,|2\rangle\Big)$$

where A is a constant and $|n\rangle$ is the n^{th} energy eigenstate.

a. Find $|\Psi(t)\rangle$ for all times $t \geq 0$.

b. Determine the expectation value of the position ($\langle x\rangle$) and momentum ($\langle p\rangle$) of the particle for all times t.

c. At time $t = T$, a measurement of the particle energy is made. Determine all possible outcomes of the measurement and their probabilities.

d. For the same particle in (c), assume that the measurement outcome is the largest of the the possible outcomes. What will the the possible measurement outcomes for the total energy of the particle for $t > T$?

Reminder: raising and lowering operators for harmonic potential are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}\,, \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$$

Solution:

(a) Normalization requires that the following condition is met:

$$\begin{aligned} 1 &= \langle\Psi(0)|\Psi(0)\rangle \\ &= |A|^2(\langle 0|0\rangle + 2\,\langle 2|0\rangle + 2\,\langle 0|2\rangle + 4\,\langle 2|2\rangle) \\ &= 5|A|^2 \end{aligned}$$

So,

$$A = \frac{1}{\sqrt{5}} \tag{1}$$

The time dependent wave function is obtained by appropriately multiplying the stationary states by their respective time dependent terms ($e^{-iE_n t/\hbar}$):

$$|\Psi(t)\rangle = \frac{1}{\sqrt{5}}\left(e^{-i\omega t/2}|0\rangle + e^{-i5\omega t/2}|2\rangle\right) \tag{2}$$

(b) The expectation values for position and momentum are:

$$\langle x\rangle = \langle\Psi(t)|\,\hat{x}\,|\Psi(t)\rangle \tag{3}$$
$$\langle p\rangle = \langle\Psi(t)|\,\hat{p}\,|\Psi(t)\rangle \tag{4}$$

These expectation values are most easily calculated by expressing $\hat{x}$ and $\hat{p}$ in terms of the ladder operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \tag{5}$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{6}$$

Looking at $\langle x\rangle$,

$$\begin{aligned}\langle x\rangle =& \langle\Psi(t)|\,\hat{x}\,|\Psi(t)\rangle\\ =&\frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\left[e^{i\omega t/2}\langle 0| + 2e^{i5\omega t/2}\langle 2|\right]\left(\hat{a}^\dagger + \hat{a}\right)\left[e^{-i\omega t/2}|0\rangle + 2e^{-i5\omega t/2}|2\rangle\right]\\ =&\frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\left[e^{i\omega t/2}\langle 0| + 2e^{i5\omega t/2}\langle 2|\right]\Big[e^{-i\omega t/2}|1\rangle + 2\sqrt{2}e^{-i5\omega t/2}|1\rangle +\\ &2\sqrt{3}e^{-i5\omega t/2}|3\rangle\Big]\end{aligned}$$

After the application of the ladder operators, it is clear that each ket is orthogonal to each bra in the calculation, so,

$$\langle x\rangle = 0 \tag{7}$$

By a similar rationale,

$$\langle p \rangle = 0 \tag{8}$$

(c) The state is a superposition of two eigenstates which correspond to two possible measurement outcomes for total energy:

$$|0\rangle \to E = \hbar\omega/2 \tag{9}$$

$$|2\rangle \to E = 5\hbar\omega/2 \tag{10}$$

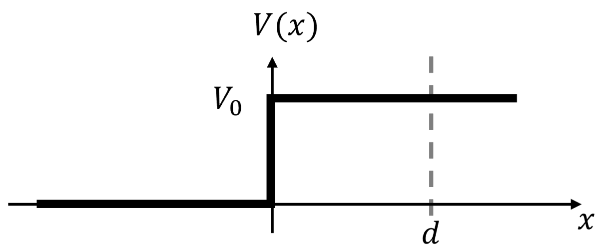
The probability of each potential is the square magnitude of the projection onto the corresponding eigenstate:

$$E = \hbar\omega/2: \quad P = |\langle 0|\Psi(T)\rangle|^2 = 1/5 \tag{11}$$

$$E = 5\hbar\omega/2: \quad P = |\langle 2|\Psi(T)\rangle|^2 = 4/5 \tag{12}$$

(d) The measurement at $t = T$ collapses the wave function to $|\Psi(t)\rangle = e^{-i5\omega t/2}|2\rangle$ for $t > T$. Therefore, all subsequent measurements of total energy will yield $E = 5\hbar\omega/2$.

(QM3) A beam of particles (i.e. a plane wave state) is traveling in the $+x$ direction from $x = -\infty$ with energy E and mass m. The beam of particles is incident on the step potential of height V_0 where $V_0 = 2E$. The wave functions cannot be normalized, so assume the incident plane wave has an amplitude A.



a. Determine the wave function for all x that describes the effect of the step potential on the incident beam.
b. Determine the position, d at which the probability density of detecting particles decreases by a factor of 10 from the probability density at $x = 0$.
c. Determine the reflection coefficient of the potential. Justify your answer.

Solution:

(a) In the region $x < 0$, the time-independent Scrhödinger equation is:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{1}$$

which has the general solution,

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \tag{2}$$

where $k = \sqrt{2mE}/\hbar$. Here the first term corresponds to the incident beam and the second is the reflected beam.

In the region $x > 0$, the time-independent Scrhödinger equation is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi \tag{3}$$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (-E)\psi \tag{4}$$

$$\tag{5}$$

which has the general solution,

$$\psi(x) = Ce^{-\kappa x} + De^{\kappa x} \tag{6}$$

where $\kappa = \sqrt{2mE}/\hbar$. Because the second term diverges as $x \to \infty$, it is unphysical and thus $D = 0$.

From the continuity of ψ and $d\psi/dx$ at $x = 0$, we have:

$$A + B = C \tag{7}$$

$$A(ik) + B(-ik) = C(-\kappa) \tag{8}$$

Using these boundary conditions, B and C can be expressed in terms of A:

$$B = A\frac{ik + \kappa}{ik - \kappa} \tag{9}$$

$$C = A\frac{2ik}{ik - \kappa} \tag{10}$$

So, the full solution is:

$$\psi(x) = A\begin{cases} e^{ikx} + \frac{ik+\kappa}{ik-\kappa}e^{-ikx}, & x \leq 0 \\ \frac{2ik}{ik-\kappa}e^{-\kappa x}, & x \geq 0 \end{cases} \tag{11}$$

(b) The probability density for detecting a particle at $x = 0$ is:

$$p(x) = |\psi(x)|^2 \tag{12}$$

At $x = 0$, the probability density is $p(0) = |C|^2$. Therefore, we need to find x_0 such that $p(x_0) = |C|^2/10$.

Using the wave function from part (a), we get the following equation:

$$\frac{|C|^2}{10} = |Ce^{-\kappa x}|^2 \tag{13}$$

Solving for x yields:

$$x = \frac{\hbar \ln 10}{\sqrt{2^3 mE}} \tag{14}$$

(c) Because the potential step is infinitely long, the particles will never tunnel through it and the reflection coefficient must be $R = 1$. This can be directly calculated from the wavefunction:

$$R = \frac{|B|^2}{|A|^2} = \left|\frac{ik + \kappa}{ik - \kappa}\right|^2 = 1 \tag{15}$$

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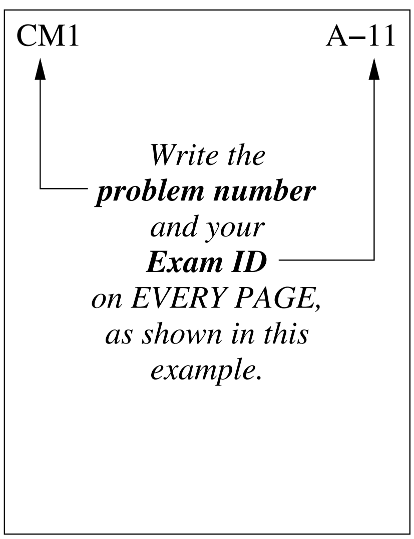
Day 3
Electricity and Magnetism



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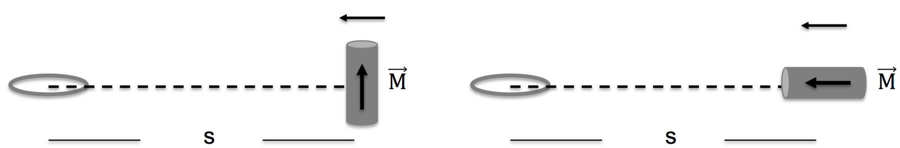
(EM1) A circular cylinder bar magnet of radius a, length l, and uniform **magnetization** $\vec{M}$ is pushed toward a circular loop of radius b and self-inductance L. The magnet remains at distances s far from the loop: $s \gg a, b, l$. The loop is a perfect conductor. Initially, the magnet is at $s \to \infty$ and there is no current in the loop.



(a) (b)

a. In the configuration shown in Figure (a), find the magnetic flux through the loop when the bar magnet is at the distance s ($s \gg a, b, l$).

b. In the configuration shown in Figure (a), what is the direction of the induced current in the loop when the bar magnet moves toward the circular loop?

c. Find the magnitude of the induced current as a function of s, when the bar magnet is moving towards the loop from infinity.

d. What is the current in the loop if the orientation of the bar magnet is different, as shown in Figure (b)?

Solution:
(a.) Given that $s \gg a, b, l$, the magnetic field by the bar magnet at the location of the loop can be evaluated as the magnetic dipole field,

$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi} \frac{3\hat{m} \cdot \hat{r} - \hat{m}}{r^3}, \tag{1}$$

where $m = \pi a^2 l M$ is the magnetic dipole moment, r is the distance of the loop to the magnetic dipole, $\hat{m}$ is the direction of the dipole, and $\hat{r}$ is the direction of the separation vector of the loop with respect to the dipole moment. In the configuration of (a), we may take $\vec{M} = M\hat{z}$, and the loop in the xy plane. It is seen $\hat{m} \perp \hat{r}$, and $r = s$, hence the magnetic field $\vec{B}_{dip}$ at the location of the loop is given by

$$\vec{B}_{dip} = -\frac{\mu_0 \pi a^2 l M}{4\pi s^3}\hat{z}. \tag{2}$$

The total flux through the loop is therefore

$$\Phi = \vec{B}_{dip} \cdot \pi b^2 \hat{z} = -\frac{\mu_0 \pi^2 b^2 a^2 l M}{4\pi s^3}, \tag{3}$$

the negative sign indicating that the dipole field is in $-\hat{z}$ direction.

(b.) As the bar magnet moves towards the loop, the flux through the loop increases, according to Lenz's law, the induced current will be in the direction that tends to reduce the (negative) flux by generating (positive) flux of the opposite sign; so the direction of the current is counter-clockwise as viewed from above the loop.

(c.) From Faraday's law, the induced current in the loop follows the relation

$$L\frac{dI}{dt} = -\frac{d\Phi}{dt}. \tag{4}$$

Integrate both sides of the equation, we arrive at

$$L(I - I_0) = -\Phi(s) + \Phi(\infty), \tag{5}$$

where $I_0 = 0$ is the initial current, and $\Phi(\infty) = 0$ is the flux through the loop initially when the bar magnet is at infinity. Therefore, we find the induced current as a function of the distance s,

$$I = \frac{\mu_0 \pi^2 b^2 a^2 l M}{4\pi L s^3}. \tag{6}$$

(d.) In the configuration of (b), the dipole field by the bar magnet is in the plane of the loop, so the magnetic flux through the loop is zero. As the bar moves toward the loop, there is no change of the magnetic flux through the loop. Therefore, there is no induced current in the loop.

(EM2) A spherical shell surrounds a central point charge at the origin. The graph represents the electric field as a function of distance from the origin. From the graph determine (and argue your answers):

(a) The approximate value of the point charge at the origin;

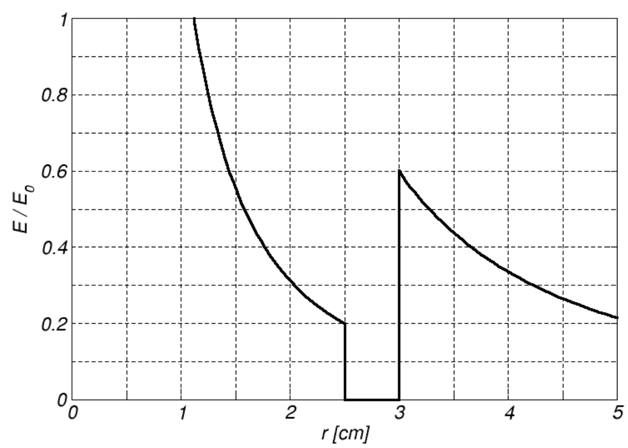
(b) whether the shell is metallic or dielectric;

(c) whether the shell is grounded or not;

(d) the *total* charge on the shell;

(e) distribution of charge on the shell, make a sketch.

The vertical scale is in terms of $E_0 = 3 \cdot 10^4$ (SI units) and the Coulomb constant $k = 1/4\pi\varepsilon_0 = 9 \cdot 10^9$ (SI units).



Solution:

Let's denote inner and outer radii of the shell as

$$r_1 = 0.025\text{m}\,, \qquad r_2 = 0.03\text{m}\,,$$

the central charge inside Q_0, and total charge on the shell is Q_s.

(a) The field inside the shell follow inverse square law,

$$E(r) = k\frac{Q_0}{r^2}$$

and we can evaluate Q_0 using any point, for example right at inner radius $r = r_1 - 0$:

$$Q_0 = \frac{0.2 E_0 r_1^2}{k} \approx 0.42 \times 10^{-9}\,\text{Coulomb}$$

(b) the shell is metallic, because the field in the shell's wall $r_1 < r < r_2$ is zero;

(c) the shell is not grounded, otherwise the potential and field outside would be zero;

(d) On the outside the field is given by the full enclosed charge (inside a Gaussian surface) that is sum of the central charge and total charge on the shell:

$$E(r) = k\frac{Q'}{r^2}\,, \qquad Q' = Q_0 + Q_s\,.$$

We evaluate the full charge by using value of the field right outside the shell:

$$Q_0 + Q_s = \frac{0.6 E_0 r_2^2}{k}\,,$$

and we'll express it in terms of the central charge:

$$\frac{Q_s}{Q_0} + 1 = \frac{0.6 r_2^2}{0.2 r_1^2} \approx 4.33\,,$$

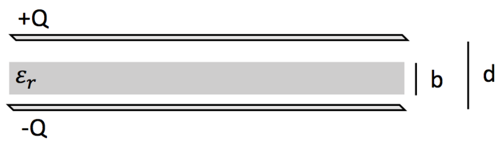
so the total charge on the shell is

$$Q_s \approx 3.33\,Q_0 = 1.38 \times 10^{-9}\,\text{Coulomb}$$

(e) charge on the shell has $-Q_0$ distributed uniformly over $4\pi r_1^2$ area on the inner surface, and $Q' = 4.33 Q_0$ uniformly spread over $4\pi r_2^2$ on the outer surface. We can write the overall charge density, including the central charge, as

$$\rho(r) = Q_0 \delta^3(\mathbf{r}) + \frac{-Q_0}{4\pi r_1^2}\delta(r - r_1) + \frac{4.33 Q_0}{4\pi r_2^2}\delta(r - r_2)$$

(EM3) A parallel capacitor is made of two metal plates of area A separated by distance d. It is charged with $\pm Q$. A dielectric slab of thickness b ($b < d$) and dielectric constant $\epsilon_r (\equiv \epsilon/\epsilon_0) > 1$ is placed in between the plates, as shown in the figure.



(a) Sketch field lines of $\vec{E}$ (electric field), $\vec{D}$ (displacement field), and $\vec{P}$ (polarization) everywhere in the capacitor.

(b) Find the capacitance C of the capacitor – ignore edge effects. What is C at the limit $b = 0$ and $b = d$, respectively?

(c) Suppose that we first charge the capacitor to $\pm Q$ using a battery **without** the slab in place. We then remove the battery and slide the slab into place. Find the work done by the electric field to polarize the slab. Is this work positive or negative?

(d) If, on the other hand, the capacitor stays connected to the battery (of fixed voltage V) all the times, is the total electric field energy in the capacitor increased or decreased after the dielectric slab is placed in the capacitor? Explain why.

Solution:

(a) Without the dielectric material, the electric field $\vec{E}$ inside the capacitor is uniform and points from the top plate to the bottom plate. The dielectric material placed inside the capacity will be polarized, and the uniform polarization $\vec{P}$ inside the material will be in the same direction of the electric field, pointing downward. As a result, surface bound charges are produced with negative/positive σ_b at the top/bottom of the material, which reduces the electric field $\vec{E}$ inside but does not change the field outside the material. The displacement field $\vec{D}$, however, is not affected by the polarization; it is uniform and points from the top plate to the bottom plate.

(b) To compute the capacitance, we will find the potential difference between the two plates, which is the path integral of the electric field. To find the electric field inside the capacitor, we first find $\vec{D}$ from Gauss's law, which only depends on $\pm Q$ at the top and bottom plates.

$$\vec{D} = -|\sigma_f|\hat{z} = -\frac{Q}{A}\hat{z}. \tag{1}$$

The electric field inside and outside the material is therefore,

$$\begin{aligned} \vec{E} &= \frac{1}{\epsilon}\vec{D} = -\frac{1}{\epsilon_r\epsilon_0}\frac{Q}{A}\hat{z} \qquad - \text{(inside material)}, \\ \vec{E} &= \frac{1}{\epsilon_0}\vec{D} = -\frac{1}{\epsilon_0}\frac{Q}{A}\hat{z} \qquad - \text{(outside material)}. \end{aligned} \tag{2}$$

The potential difference is therefore,

$$\Delta V = -\int_0^d \vec{E}\cdot\hat{z}dz = \frac{1}{\epsilon_r\epsilon_0}\frac{Q}{A}b + \frac{1}{\epsilon_0}\frac{Q}{A}(d-b) = \frac{Q}{\epsilon_0 A}\left[\frac{b}{\epsilon_r} + (d-b)\right]. \tag{3}$$

And the capacitance is therefore

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{\frac{b}{\epsilon_r} + (d-b)}. \tag{4}$$

We can check the solution at the two limits; when $b = 0$, $C = (\epsilon_0 A)/d$, exactly the capacitance without the material. When $b = d$, $C = (\epsilon A)/d$, as expected.

(c) The work done by the electric field is the difference between the electric field energy before and after the slab is placed in. The total electric field energy inside the capacitor is given by

$$W = \frac{1}{2}\frac{Q^2}{C}. \tag{5}$$

So the difference in the energy is

$$\Delta W = \frac{1}{2}Q^2\left(\frac{1}{C} - \frac{1}{C_0}\right) = \frac{1}{2}\frac{Q^2 b}{\epsilon_0 A}\left(\frac{1}{\epsilon_r} - 1\right) < 0. \tag{6}$$

The change in the energy is negative, so the electric field does positive work to polarize the material – positive/negative bound charges moved in the same/opposite direction of the electric field!

(d) If on the other hand, the capacitor is connected to the battery with a constant voltage V, the electric field energy is given by

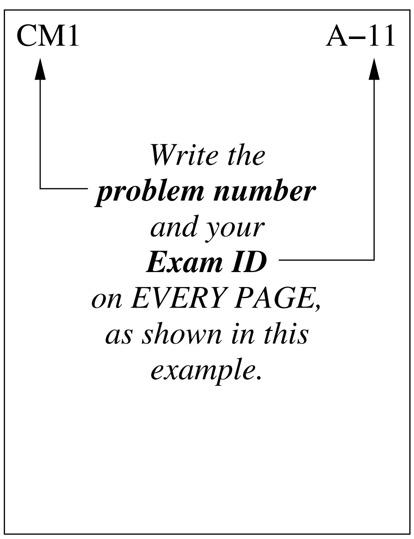
$$W = \frac{1}{2}CV^2. \tag{7}$$

After the slab is placed in, the capacitance increases, $C > C_0$, so the total energy increases. Electric field still does positive work to polarize the material, but the battery is doing more work against the charges on the plates to transport more free charges to the capacitor, and increases the total electric field energy in the capacitor.

Department of Physics
Montana State University

Qualifying Exam
August, 2024

Day 4
Statistical and Thermal Physics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(ST1) Our goal is to find the equation of state of a white dwarf, which can be well approximated as a zero-temperature Fermi gas of non-interacting electrons. The equation of state is the relation between pressure P and the electron number density n_e, which follows a power-law relation $P \propto n_e^\Gamma$. Our goal is to find the value of Γ. Suppose the white dwarf has volume V and N electrons, so $n_e = N/V$. The electrons can be approximated as non-relativistic particles with its energy ϵ and momentum $p = |\boldsymbol{p}|$ related by $\epsilon = p^2/(2m_e)$.

(a). The Fermi energy ϵ_F is the maximum energy of an occupied electron state when the system is in its ground state. Compute ϵ_F in terms of $n_e = N/V$. [Hints: The spherical geometry of the star plays no role here, so you can use any geometry you like for the volume V – or none at all. If you are unable to do the exact calculation, you may try to estimate the answer based on the uncertainty principle. This will enable you to answer the rest of the problem.]

(b). Find E_t, the total energy of the system, and from it the averaged energy per particle E_t/N. How does E_t/N scale with the volume V?

(c). Find the power-law relation between pressure P vs density n_e, and the value of Γ. [Hint: You might find it convenient to find P in terms of E_t and V through the first law of thermodynamics.]

(d). Argue why we can focus only on electrons and ignore the proton/neutron contribution to the pressure.

Solution:

(a). The number density of particles over phase space can be written as

$$n(p, x)d^3pd^3x = fg_s\frac{4\pi}{h^3}p^2dpdV, \tag{1}$$

where f is the occupation number per state and g_s is the degeneracy of each state ($g_s = 2$ for spin-1/2 electrons). For zero-temperature Fermi gas, the occupation is simply given by

$$f = \begin{cases} 1, & \text{for } p \leq p_F, \\ 0, & \text{for } p > p_F, \end{cases} \tag{2}$$

where p_F is the Fermi momentum and for non-relativistic electrons, it is related to the Fermi energy as

$$\epsilon_F = \frac{p_F^2}{2m_e}. \tag{3}$$

Integrating over the phase space should give us the total number of electrons in the system, so we have the relation

$$\begin{aligned} N &= \int n(p,x) d^3p d^3x = 4\pi g_s \frac{V}{h^3} \int_0^{p_F} p^2 dp, \\ &= \frac{4\pi}{3} g_s \frac{V p_F^3}{h^3} = \frac{4\pi}{3} g_s \frac{V(2m_e\epsilon_F)^{3/2}}{h^3}. \end{aligned} \tag{4}$$

Thus,

$$\epsilon_F = \frac{h^2}{2m_e}\left(\frac{3}{4\pi g_s} n_e\right)^{2/3} = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2}{g_s} n_e\right)^{2/3}. \tag{5}$$

Or in terms of p_F,

$$p_F = h\left(\frac{3}{4\pi g_s} n_e\right)^{1/3} = \hbar\left(\frac{6\pi^2}{g_s} n_e\right)^{1/3}. \tag{6}$$

Or simply using the uncertainty principle,

$$p_F \sim \frac{h}{\Delta x} \simeq h n_e^{1/3}, \tag{7}$$

where Δx is the typical spacing between electrons, $N(\Delta x)^3 \sim V$, and $\epsilon_F = p_F^2/(2m_e) \propto n_e^{2/3}$.

(b). The total energy

$$\begin{aligned} E_t &= \int n(p,x) \frac{p^2}{2m_e} d^3p d^3x = \frac{4\pi g_s V}{2m_e h^3} \int_0^{p_F} p^4 dp, \\ &= \frac{4\pi g_s V}{5h^3} p_F^3 \frac{p_F^2}{2m_e}, \\ &= \frac{3}{5} N \frac{p_F^2}{2m_e} = \frac{3}{5} N \epsilon_F. \end{aligned} \tag{8}$$

The averaged energy per electron is

$$\frac{E_t}{N} = \frac{3}{5}\epsilon_F \propto V^{-2/3}. \tag{9}$$

Or an order of magnitude estimation, $E_t/N \sim \epsilon_F \propto V^{-2/3}$.

(c). The most convenient way to get the pressure is to use thermodynamic identity

$$P = -\frac{\partial E_t}{\partial V} = -\frac{2}{3}\frac{E_t}{V} = \frac{2}{5}n_e \epsilon_F \propto n_e^{5/3}, \tag{10}$$

so $\Gamma = 5/3$.

Or from estimation, as long as one notices $P \sim n_e \epsilon_F$, it would also lead to the correct answer $P \propto n_e^{5/3}$.

(d). Note $P \sim n_e \epsilon_F \propto m_e^{-1}$. Since proton mass $m_p \gg m_e$, its contribution to the degenerate pressure is smaller by a factor of m_e/m_p.

(ST2) A system of N non-interacting magnetic dipole moments in external magnetic field has the energy

$$E^{mag} = -\sum_{i=1}^{N} \boldsymbol{\mu}_i \cdot \mathbf{B}\,,$$

where all magnetic moments have the same magnitude $|\boldsymbol{\mu}_i| = \mu$, but can point in arbitrary direction on the unit sphere. Treating magnetic moments as classical, take $\boldsymbol{\mu}_i \cdot \mathbf{B} = \mu B \cos\theta_i$ where $\theta_i \in [0, \pi]$ is the angle relative to the direction of the external field $\mathbf{B}$.

(a) Calculate the canonical partition function of a single magnetic moment in magnetic field $Z_1^{mag}(T, B)$, in equilibrium with thermostat T, and use it to find partition function of all N moments $Z_N^{mag}(T, B)$; (You can use limit $k_B T \gg \mu B$ to simplify your calculations and answers);

(b) Find the contribution from magnetic degrees of freedom to the entropy $S(T, B)$ in the limit $k_B T \gg \mu B$, leave only the first non-trivial term; assume the entropy of the system without magnetic degrees of freedom is known $S(T, 0) = S_0(T)$;

(c) Show that during an adiabatic process when magnetic field is reduced to zero, $B \to 0$, the magnetic material is cooled.

Solution:

(a) Partition function of one magnetic moment is obtained by integrating over all configurations of the system (orientation angles of the moment in this case), with the usual exponential weight factor:

$$Z_1^{mag} = \iint \frac{d\Omega}{4\pi} e^{-E^{mag}/T} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \; e^{\mu B \cos\theta/T} = \frac{\sinh(\mu B/T)}{(\mu B/T)}$$

The partition function for the N independent moments is

$$Z_N^{mag} = [Z_1^{mag}]^N \approx \left[1 + \frac{1}{3!}\left(\frac{\mu B}{T}\right)^2\right]^N$$

where we used Taylor expansion

$$\sinh x = x + \frac{x^3}{3!} + \dots$$

Note that if one doesn't remember hyperbolic functions, one can use the approximate limit and calculate Z_1^{mag} integral with $u = \cos\theta$ substitution:

$$\begin{aligned} Z_1^{mag} &= \frac{1}{2}\int_0^{\pi} \sin\theta d\theta \; e^{\mu B\cos\theta/T} = \frac{1}{2}\int_{-1}^{+1} du \; e^{(\mu B/T)u} \\ &\approx \frac{1}{2}\int_{-1}^{+1} du \; \left(1 + \frac{\mu B}{T}u + \frac{1}{2!}\frac{\mu^2 B^2}{T^2}u^2 + \dots\right) \\ &= 1 + \frac{1}{3!}\frac{\mu^2 B^2}{T^2} + \dots \end{aligned}$$

(b) The free energy for magnetic degrees of freedom is

$$F^{mag} = -T\ln Z_N^{mag} \approx -N\frac{\mu^2 B^2}{6T}$$

and the entropy is

$$S^{mag} = -\frac{\partial F^{mag}}{\partial T} = -\frac{N}{6}\frac{\mu^2 B^2}{T^2}$$

with total entropy being

$$S(T,B) = S_0(T) - \frac{N}{6}\frac{\mu^2 B^2}{T^2}$$

Note, magnetic field reduces entropy by aligning magnetic moments into an ordered state.

(c) Consider now reducing field to zero adiabatically. In an adiabatic process the entropy is constant: $S(T_i, B) = S(T_f, 0) \quad \Rightarrow$

$$S_0(T_i) - \frac{N}{6}\frac{\mu^2 B^2}{T_i^2} = S_0(T_f) \qquad \Rightarrow \qquad S_0(T_f) - S_0(T_i) = -\frac{N}{6}\frac{\mu^2 B^2}{T_i^2}$$

The difference is negative, $S_0(T_f) < S_0(T_i)$. Since entropy is an increasing function of temperature, this implies $T_f < T_i$.

(ST3) An air conditioner works by circulating a working fluid, which we can approximate as an ideal gas, through a closed cycle of four steps: A–D. In step A the fluid is at the same temperature as the indoor air, T_{in}, and *gains heat* from it; this *removes* heat from the air. In step B, the fluid is *adiabatically* compressed at entropy S_B up to the temperature of the outside air: $T_{\text{out}} > T_{\text{in}}$. The working fluid then exchanges heat with the outside air at T_{out} (step C), and finally is adiabatically returned (step D), at entropy S_D, to its original state at T_{in}, and the same volume with which it began the cycle.

a. Draw a diagram in T *vs.* S space of the working fluid undergoing one complete cycle. Label each of the steps A–D described above on the diagram. State which step (or steps) require(s) a motor to **do positive work** on the working fluid.

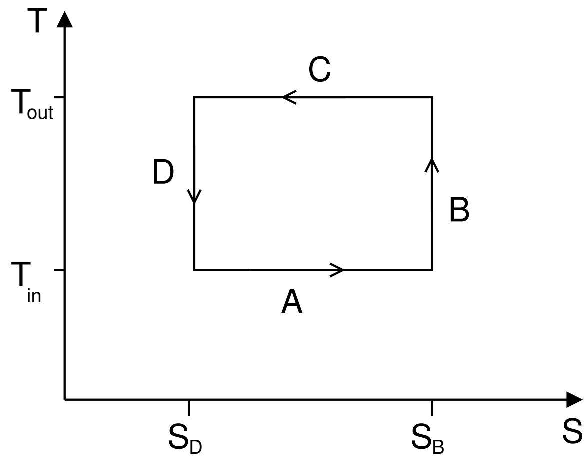
b. In terms of T_{in}, T_{out}, S_B and S_D, compute the **heat** removed from the indoor air in step A.

c. Compute the **net work** done by the motor on the fluid over one complete cycle. Assume it works perfectly by recovering all *the work done on it by the fluid*.

d. Outside is a toasty $T_{\text{out}} = 35°$, while indoors is kept at a pleasant $T_{\text{in}} = 20°$. It is a perfect system (i.e. part c.) in which the motor draws an averaged power of 1000 W. At approximately what average rate is the air conditioner removing heat from the indoor air?

Solution:

a. The cycle consists of two isothermal legs, A and C, and two adiabatic legs, B and D. The cycle thus forms a rectangle in T *vs.* S space (see below). Leg A goes along T_{in}, and since heat is added to it, $S_B > S_D$. The cycle proceeds counterclockwise — the sense opposite of the traditional Carnot heat engine.



For an ideal gas undergoing isothermal heating, $dE = C_v dT = 0$, so

$$dS \;=\; \frac{p}{T}\,dV \;=\; NR\,\frac{dV}{V} \quad . \tag{1}$$

Therefore compression, $dV < 0$, is accompanied by an entropy decrease, $dS < 0$, as in leg C. For an adiabatic process, $dS = 0$, so the work done on the fluid is

$$dW \;=\; -\,pdV \;=\; dE \;=\; C_v\,dT \quad . \tag{2}$$

This means compression generates increasing T, as in leg B. Thus positive work is being done on the working fluid, by some motor, **along legs B and C.**

b. Differential heating is $dQ = T\,dS$, so the heat exchanged along leg A is found from the integral

$$\Delta Q_A \;=\; \int\limits_A T\,dS \;=\; T_{\rm in}\int\limits_A dS \;=\; T_{\rm in}\,(S_B - S_D) \quad . \tag{3}$$

c. Since the working fluid returns to its initial state, $\oint dE = 0$ after a complete cycle, and the net work done *on* the fluid is

$$\begin{aligned} W \;&=\; -\oint p\,dV \;=\; -\oint T\,dS \;=\; -\int\limits_A T\,dS \;-\; \int\limits_C T\,dS \\ &=\; -T_{\rm in}\,(S_B - S_D) \;-\; T_{\rm out}\,(S_D - S_B) \qquad (4) \\ &=\; (T_{\rm out} - T_{\rm in})\,(S_B - S_D) \quad , \end{aligned}$$

which is the area inside the rectangle on the T *vs.* S diagram.

d. Over a single cycle the ratio of heat removal to work

$$\frac{\Delta Q_A}{W} = \frac{T_{\rm in}}{T_{\rm out} - T_{\rm in}} \ , \tag{5}$$

depends only on the temperatures of legs A and C. Averaging over whole cycles gives the heating rate

$$\begin{aligned} \left\langle \frac{dQ_A}{dt} \right\rangle &= \frac{T_{\rm in}}{T_{\rm out} - T_{\rm in}} \left\langle \frac{dW}{dt} \right\rangle = \frac{293\,{\rm K}}{15\,{\rm K}} \times 1000\,{\rm W} \\ &\simeq 20,000\,{\rm W} \ , \end{aligned} \tag{6}$$

for the values quoted. To do this one must convert $T_{\rm in} = 20^\circ\,{\rm C} = 293\,{\rm K}$.