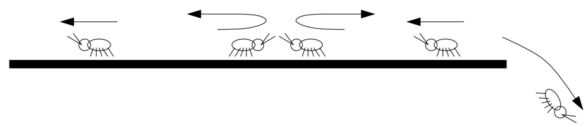
Ants on a stick



One hundred ants are dropped on a meter stick. Each ant is traveling either to the left or the right with constant speed 1 meter per minute. When two ants meet, they bounce off each other and reverse direction. When an ant reaches an end of the stick, it falls off.

At some point all the ants will have fallen off. The time at which this happens will depend on the initial configuration of the ants.

Questions:

(a) over ALL possible initial configurations, what is the longest amount of time that you would need to wait to guarantee that the stick has no more ants?

(b) a slightly more technical task: calculate the *average* time it takes for arbitrary initial number of ants N to fall off the stick.

Answer of problem **Ants on a stick**

Let L be the length of the stick, and v is the speed of ants.

(a) during each encounter=‘collision’, ants exchange velocities, and if we forget about ants being different, we could say that the ants simply go through each other and move independently of each other. Then the longest possible time it takes for all ants to fall off the stick would be when there is an ant at one end of the stick, and it goes towards the opposite end. Then it takes time

$$T_{max} = \frac{L}{v} = \boxed{1\ minute}$$

(b) To classify all configurations we note that we can ignore wich direction, left or right, ants are moving originally, since ants can “go through each other” and thus only their original direction of motion and distance to the corresponding end matter. So then we can consider only configurations where all ants are moving in one direction (say left) and only think about their distribution along the length of the stick.

What matters is the position of the last ant, the one most to the right. Lets say it is at point x on the stick of length L. It can be any of the N ants. All other $N-1$ ants are randomly placed left of x in the interval $[0, x]$. Probability of this configuration is

$$P(x)dx = N \left(\frac{x}{L}\right)^{N-1} \frac{dx}{L} \qquad\qquad check: \qquad \int_0^L P(x)dx = 1$$

The average time it takes for all ants to fall off is average of ‘escape’ times for each configuration $T_x = x/v$,

$$\langle T \rangle = \int_0^L \frac{x}{v} P(x)dx = N \int_0^L \frac{x}{v} \left(\frac{x}{L}\right)^{N-1} \frac{dx}{L} = \boxed{\frac{L}{v}\frac{N}{N+1}}$$

For one ant the average time is $L/2v$ since its average position on the stick is $L/2$. For a lot of ants the average time is close to L/v since most of the configurations will have ants close to the ends of the stick when they have to traverse stick’s entire length.

For 100 1m/s ants on the meter, the average time is 59.4 seconds.