Ice Age Hockey

The Earth is covered with ice. The surface of ice is perpendicular to the direction of a hanging plumb bob at all points (*i.e.* normal to the "effective gravity" locally).

Somewhere in Yellowstone region...

Sid the sloth took away THE nut from Scrat the saber-toothed squirrel, and tries to pass the nut to Manny the mammoth, who is 5 km to the North from Sid and Scrat. Sid gives the nut a push and the nut starts to slide in northern direction with speed $v = 1\ m/s$ (no friction). How long does Manny have to wait for the nut to arrive? Can Scrat get the nut back, if he can only run along constant latitude? Where should he place himself and how long he has to wait?

Assume that the typical size of nut's trajectory is small compared to Earth size.

Answer of problem **Ice Age Hockey**

The nut on the ice surface will be traveling in a circle, all due to the Coriolis Effect!

By construction, the normal force from the ice exactly cancels all effects of the gravitational and centrifugal forces in the rotating frame of the earth (because the plumb bob hangs in the direction of the "effective gravity" force, which is the sum of the gravitational and centrifugal forces. We therefore need only concern ourselves with the Coriolis force. This force equals

$$\mathbf{F}_{cor} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

Let the angle down from the north pole be $\theta \approx 45°$ for Yellowstone (we assume the circle is small enough so that θ is essentially constant throughout the motion). Introduce local coordinates (x, y, z): z is normal to the ice surface pointing up, y - points to the north, x - points to the east. Since we are interested only in motion in the xy-plane, we need just the z-projection of rotation:

$$\boldsymbol{\omega}_z = \omega \mathbf{z} \cos\theta$$

The component of the Coriolis force that points horizontally along the surface is

$$\mathbf{f} = -2m\omega\cos\theta \mathbf{z} \times \mathbf{v}$$

and it is perpendicular to the direction of motion! (The vertical component of the Coriolis force will simply modify the required normal force and is not interesting.) Because this force is perpendicular to the direction of motion, v does not change. This is the same as the Lorentz force on a moving charge in magnetic field!

If we introduce complex velocity $w = v_x + iv_y$, the vector equation of motion in xy-plane can be written as a single scalar equation:

$$m\dot{\mathbf{v}} = -2m\omega\cos\theta\, \mathbf{z} \times \mathbf{v} \qquad \Rightarrow \qquad w = v_x + iv_y$$

$$\Rightarrow \qquad \dot{w} = -i(2\omega\cos\theta)\, w \qquad \Rightarrow \qquad \boxed{w(t) = w_0 e^{-i\Omega\, t}}$$

which describes a clockwise rotation with frequency

$$\boxed{\Omega = 2\omega\cos\theta}$$

For $v = 1m/s$, using $\omega = 2\pi/1\,day = 7.3 \cdot 10^{-5}\ sec^{-1}$, the radius of motion $r \approx 10\ km$.

So the sabre-tooth squirrel should go 20 km to the east to get the nut. The nut will arrive there in

$$T = \frac{\pi}{\Omega} = \frac{\pi}{2\omega\cos\theta} = \frac{1\ day}{4\cos 45°} \approx 8.5\ hours$$