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A mathematical trivium

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The standard of mathematical culture is falling; both undergraduate and postgraduate students leaving our colleges, including the Mechanics and Mathematics Faculty of Moscow State University, are becoming more ignorant than the professors and teachers. What is the reason for this abnormal phenomenon? Under normal conditions students know their subject better than their professors in accordance with the general principle of the diffusion of knowledge: new knowledge prevails not because it is taught by old men, but because new generations come along who know it.

Among the causes of this abnormal situation I would like to single out those for which we ourselves are responsible, so that we can try to correct what is within our power. One such cause, I believe, is our system of examinations, which is specially designed for the systematic production of rejects, that is, pseudo-pupils who learn mathematics like Marxism: they cram themselves with formulae and rote-learning of answers to the most frequent examination questions.

How can the standard of training of a mathematician be measured? Neither a list of courses nor their syllabuses determine the standard. The only way to determine what we have actually taught our students is to list the problems which they should be able to solve as a result of their instruction.

I am not talking about difficult kinds of problems, but about the simple questions which form the strictly essential minimum. There need not necessarily be many of these problems, but we must insist that the students are able to solve them. I.E. Tamm used to tell the story that having fallen into the hands of the Makhnovtsy during the Civil War, he said under interrogation that he taught in the physics and mathematics faculty. He owed his life to the fact that he could solve a problem in the theory of series, which was put to him as a test of his veracity. Our students should be prepared for such ordeals!

Throughout the world a mathematics examination consists of the written solution of problems. The written character of the test is everywhere considered just as much a necessary sign of a democratic society as the choice between several candidates in an election. In fact, in an oral examination the student is completely defenceless. While conducting examinations in the department of differential equations in the Mechanics and Mathematics Faculty of Moscow State University, I have overheard examiners at a nearby table failing students who gave immaculate answers (which perhaps exceeded the level of comprehension of the teacher). Cases have also been known when they have failed a student on purpose (sometimes I have saved the situation by entering the examination room).

Written work is a document and the examiner is perforce more objective in marking it (particularly if the work to be marked is anonymous, as it should be).

There is another not unimportant advantage of written examinations: the problems are preserved and can be published or passed on to students of the next course in preparation for their examination. In addition, these problems determine both the standard of the course and the standard of the teacher who has compiled them. His strong and weak points can be seen at once, and specialists can immediately assess the teacher both in respect of what he wants to teach the students and what he has succeeded in teaching them.

Incidentally, in France the problems in the *Concours général*, common to the whole country and roughly equivalent to our Olympiad, are compiled by teachers sending their problems to Paris where the best are chosen. The Ministry obtains objective data about the standard of its teachers by comparing firstly the problems set, and secondly the results of their pupils. With us, however, teachers are assessed as you know by indications such as their external appearance, quickness of speech, and ideological "correctness".

It is not surprising that other countries are unwilling to recognize our diplomas (in future I think this will even extend to diplomas in mathematics). Assessments obtained from oral examinations that leave no records cannot be objectively compared with anything else and have an extremely vague and relative weight, wholly dependent on the real standard of teaching and the demands made in a given college. With the same syllabus and marks the knowledge and ability of the graduates may vary (in some sense) by a factor of ten. Besides, an oral examination can be far more easily falsified (this has even happened with us at the Mechanics and Mathematics Faculty of Moscow State University, where, as a blind teacher once said, a good mark must be given to a student whose answer is "very close to the textbook", even if he cannot answer a single question).

The essence and the shortcomings of our system of mathematical education have been brilliantly described by Richard Feynman in his memoirs (*Surely you're joking, Mr Feynman* (Norton, New York 1984), in the chapter on physics education in Brazil, a Russian translation of which was published in Uspekhi Fizicheskikh Nauk **148**:3 (1986)).

In Feynman's words, these students understand nothing, but never ask questions, so that they appear to understand everything. If anybody begins to ask questions, he is quickly put in his place, as he is wasting the time of the lecturer dictating his lecture and the students copying it down. The result is that no one can apply anything they have been taught to even a single example. The examinations too (dogmatic like ours: state the definition, state the theorem) are always successfully passed. The students reach a state of "self-propagating pseudo-education" and can teach future generations in the same way. But all this activity is completely senseless, and in fact our output of specialists is to a significant extent a fraud, an illusion and a sham: these so-called specialists are not in a position to solve the simplest problems, and do not possess the rudiments of their trade.

Thus, *to put an end to this spurious enhancement of the results, we must specify not a list of theorems, but a collection of problems which students should be able to solve. These lists of problems must be published annually* (*I think there should be ten problems for each one-semester course*). Then we shall see what we really teach the students and how far we are successful. So that the students learn to apply their knowledge, *all examinations must be written examinations.*

Naturally the problems will vary from college to college and from year to year. Then the standard of different teachers and the output in different years can be compared. A student who takes much more than five minutes to calculate the mean of $\sin^{100} x$ with 10% accuracy has no mastery of mathematics, even if he has studied non-standard analysis, universal algebra, supermanifolds, or embedding theorems.

The compilation of model problems is a laborious job, but I think it must be done. As an attempt I give below a list of one hundred problems forming a mathematical minimum for a physics student. Model problems (unlike syllabuses) are not uniquely defined, and many will probably not agree with me. Nonetheless I assume that it is necessary to begin to determine mathematical standards by means of written examinations and model problems. It is to be hoped that in the future students will receive model problems for each course at the beginning of each semester, and oral examinations for which the students cram by heart will become a thing of the past.

1. Sketch the graph of the derivative and the graph of the integral of a function given by a free-hand graph.
2. Find the limit

$$\lim_{x\to 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}.$$

3. Find the critical values and critical points of the mapping $z \mapsto z^2 + 2\bar{z}$ (sketch the answer).
4. Calculate the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}.$$

5. Calculate the 100th derivative of the function

$$\frac{1}{x^2+3x+2}$$

at $x = 0$ with 10% relative error.

6. In the (x, y)-plane sketch the curve given parametrically by

$$x = 2t - 4t^3, \quad y = t^2 - 3t^4.$$

7. How many normals to an ellipse can be drawn from a given point of the plane? Find the region in which the number of normals is maximal.

8. How many maxima, minima, and saddle points does the function $x^4+y^4+z^4+u^4+v^4$ have on the surface $x+ \ldots +v = 0$, $x^2+ \ldots + v^2 = 1$, $x^3+ \ldots + v^3 = C$?

9. Does every positive polynomial in two real variables attain its lower bound in the plane?

10. Investigate the asymptotic behaviour of the solutions y of the equation $x^5+x^2y^2 = y^6$ that tend to zero as $x \to 0$.

11. Investigate the convergence of the integral

$$\iint\limits_{-\infty}^{+\infty} \frac{dx\,dy}{1+x^4y^4}.$$

12. Find the flux of the vector field $\vec{r}/r^3$ through the surface

$$(x-1)^2 + y^2 + z^2 = 2.$$

13. Calculate with 5% relative error

$$\int_1^{10} x^x\,dx.$$

14. Calculate with at most 10% relative error

$$\int_{-\infty}^{\infty} (x^4 + 4x + 4)^{-100}dx.$$

15. Calculate with 10% relative error

$$\int_{-\infty}^{\infty} \cos(100(x^4 - x))\,dx.$$

16. What fraction of the volume of a 5-dimensional cube is the volume of the inscribed sphere? What fraction is it of a 10-dimensional cube?

17. Find the distance of the centre of gravity of a uniform 100-dimensional solid hemisphere of radius 1 from the centre of the sphere with 10% relative error.

18. Calculate

$$\int \ldots \int e^{-\sum\limits_{1 \leqslant i \leqslant j \leqslant n} x_i x_j} dx_1 \ldots dx_n.$$

19. Investigate the path of a light ray in a plane medium with refractive index $n(y) = y^4 - y^2 + 1$, using Snell's law $n(y) \sin\alpha = \text{const}$, where α is the angle made by the ray with the y-axis.

20. Find the derivative of the solution of the equation $\ddot{x} = x + A\dot{x}^2$, with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, with respect to the parameter A for $A = 0$.

21. Find the derivative of the solution of the equation $\ddot{x} = \dot{x}^2 + x^3$ with initial condition $x(0) = 0$, $\dot{x}(0) = A$ with respect to A for $A = 0$.

22. Investigate the boundary of the domain of stability ($\max \operatorname{Re} \lambda_j < 0$) in the space of coefficients of the equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$.

23. Solve the quasi-homogeneous equation

$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

24. Solve the quasi-homogeneous equation

$$\ddot{x} = x^5 + x^2\dot{x}.$$

25. Can an asymptotically stable equilibrium position become unstable in the Lyapunov sense under linearization?

26. Investigate the behaviour as $t \to +\infty$ of solutions of the systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2\sin y - y - x, \end{cases} \quad \begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^3 - x^2 - \varepsilon y, \end{cases}$$

where $\varepsilon \ll 1$.

27. Sketch the images of the solutions of the equation

$$\ddot{x} = F(x) - k\dot{x}, \quad F = -dU/dx,$$

in the (x, E)-plane, where $E = \dot{x}^2/2 + U(x)$, near non-degenerate critical points of the potential U.

28. Sketch the phase portrait and investigate its variation under variation of the small complex parameter ε:

$$\dot{z} = \varepsilon z - (1 + i) z |z|^2 + \bar{z}^4.$$

29. A charge moves with velocity 1 in a plane under the action of a strong magnetic field $B(x, y)$ perpendicular to the plane. To which side will the centre of the Larmor neighbourhood drift? Calculate the velocity of this drift (to a first approximation). [Mathematically, this concerns the curves of curvature NB as $N \to \infty$.]

30. Find the sum of the indexes of the singular points other than zero of the vector field $z\bar{z}^2 + z^4 + 2\bar{z}^4$.

31. Find the index of the singular point 0 of the vector field with components

$$(x^4 + y^4 + z^4, \quad x^3y - xy^3, \quad xyz^2).$$

32. Find the index of the singular point 0 of the vector field

$$\operatorname{grad}(xy + yz + xz).$$

33. Find the linking coefficient of the phase trajectories of the equation of small oscillations $\ddot{x} = -4x$, $\ddot{y} = -9y$ on a level surface of the total energy.

34. Investigate the singular points on the curve $y = x^3$ in the projective plane.

35. Sketch the geodesics on the surface

$$(x^2 + y^2 - 2)^2 + z^2 = 1.$$

36. Sketch the evolvent of the cubic parabola $y = x^3$ (the evolvent is the locus of the points $\vec{r}(s) + (c - s)\dot{\vec{r}}(s)$, where s is the arc-length of the curve $\vec{r}(s)$ and c is a constant).

37. Prove that in Euclidean space the surfaces

$$((A - \lambda E)^{-1} x, x) = 1$$

passing through the point x and corresponding to different values of λ are pairwise orthogonal (A is a symmetric operator without multiple eigenvalues).

38. Calculate the integral of the Gaussian curvature of the surface

$$z^4 + (x^2 + y^2 - 1)(2x^2 + 3y^2 - 1) = 0.$$

39. Calculate the Gauss integral

$$\oint\!\!\oint \frac{(d\vec{A}, d\vec{B}, \vec{A} - \vec{B})}{|\vec{A} - \vec{B}|^3},$$

where $\vec{A}$ runs along the curve $x = \cos\alpha$, $y = \sin\alpha$, $z = 0$, and $\vec{B}$ along the curve $x = 2\cos^2\beta$, $y = \frac{1}{2}\sin\beta$, $z = \sin 2\beta$.

40. Find the parallel displacement of a vector pointing north at Leningrad (latitude 60°) from west to east along a closed parallel.

41. Find the geodesic curvature of the line $y = 1$ in the upper half-plane with the Lobachevskii–Poincaré metric

$$ds^2 = (dx^2 + dy^2)/y^2.$$

42. Do the medians of a triangle meet in a single point in the Lobachevskii plane? What about the altitudes?

43. Find the Betti numbers of the surface $x_1^2+ \dots +x_k^2-y_1^2- \dots -y_l^2 = 1$ and the set $x_1^2+ \dots +x_k^2 \leqslant 1+y_1^2+ \dots +y_l^2$ in a $(k+l)$-dimensional linear space.

44. Find the Betti numbers of the surface $x^2+y^2 = 1+z^2$ in three-dimensional projective space. The same for the surfaces $z = xy$, $z = x^2$, $z^2 = x^2+y^2$.

45. Find the self-intersection index of the surface $x^4+y^4 = 1$ in the projective plane CP^2.

46. Map the interior of the unit disc conformally onto the first quadrant.

47. Map the exterior of the disc conformally onto the exterior of a given ellipse.

48. Map the half-plane without a segment perpendicular to its boundary conformally onto the half-plane.

49. Calculate

$$\oint_{|z|=2} \frac{dz}{\sqrt{1+z^{10}}}.$$

50. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\,dx.$$

51. Calculate the integral

$$\int_{-\infty}^{\infty} e^{ikx}\,\frac{1-e^x}{1+e^x}\,dx.$$

52. Calculate the first term of the asymptotic expression as $k \to \infty$ of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{\sqrt{1+x^{2n}}}.$$

53. Investigate the singular points of the differential form $dt = dx/y$ on the compact Riemann surface $y^2/2+U(x) = E$, where U is a polynomial and E is not a critical value.

54. $\ddot{x} = 3x-x^3-1$. In which of the potential wells is the period of oscillation greater (in the shallower or the deeper) with equal values of the total energy?

55. Investigate topologically the Riemann surface of the function

$$w = \arctan z.$$

56. How many handles has the Riemann surface of the function

$$w = \sqrt{1+z^n}?$$

57. Find the dimension of the solution space of the problem $\partial u/\partial \bar{z} = \delta(z-i)$ for $\operatorname{Im} z \geqslant 0$, $\operatorname{Im} u(z) = 0$ for $\operatorname{Im} z = 0$, $u \to 0$ as $z \to \infty$.

58. Find the dimension of the solution space of the problem $\partial u/\partial \bar{z} = a\delta(z-i)+b\delta(z+i)$ for $|z| \leqslant 2$, $\operatorname{Im} u = 0$ for $|z| = 2$.

59. Investigate the existence and uniqueness of the solution of the problem $yu_x = xu_y$, $u|_{x=1} = \cos y$ in a neighbourhood of the point $(1, y_0)$.

60. Is there a solution of the Cauchy problem

$$x(x^2+y^2)\frac{\partial u}{\partial x}+y^3\frac{\partial u}{\partial y} = 0, \quad u|_{y=0} = 1$$

in a neighbourhood of the point $(x_0, 0)$ of the x-axis? Is it unique?

61. What is the largest value of t for which the solution of the problem

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} = \sin x, \quad u|_{t=0} = 0,$$

can be extended to the interval $[0, t)$?

62. Find all solutions of the equation $y\ \partial u/\partial x - \sin x\ \partial u/\partial y = u^2$ in a neighbourhood of the point (0, 0).

63. Is there a solution of the Cauchy problem $y\ \partial u/\partial x + \sin x\ \partial u/\partial y = y$, $u|_{x=0} = y^4$ on the whole (x, y) plane? Is it unique?

64. Does the Cauchy problem $u|_{y=x^2} = 1$, $(\nabla u)^2 = 1$ have a smooth solution in the domain $y \geqslant x^2$? In the domain $y \leqslant x^2$?

65. Find the mean value of the function $\ln r$ on the circle $(x-a)^2+(y-b)^2 = R^2$ (of the function $1/r$ on the sphere).

66. Solve the Dirichlet problem

$$\begin{array}{lll} \Delta u = 0 & \text{for} & x^2+y^2<1; \\ u = 1 & \text{for} & x^2+y^2=1,\ y>0; \\ u = -1 & \text{for} & x^2+y^2=1,\ y<0. \end{array}$$

67. What is the dimension of the space of solutions continuous on $x^2+y^2 \geqslant 1$ of the problem

$$\Delta u = 0 \quad \text{for} \quad x^2+y^2>1, \quad \partial u/\partial n = 0 \quad \text{for} \quad x^2+y^2=1?$$

68. Find

$$\inf \iint\limits_{x^2+y^2\leqslant 1} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\, dy$$

for C^∞-functions u that vanish at 0 and are equal to 1 on $x^2+y^2 = 1$.

69. Prove that the solid angle based on a given closed contour is a function of the vertex of the angle that is harmonic outside the contour.

70. Calculate the mean value of the solid angle by which the disc $x^2+y^2 \leqslant 1$ lying in the plane $z = 0$ is seen from points of the sphere $x^2+y^2+(z-2)^2 = 1$.

71. Calculate the charge density on the conducting boundary $x^2+y^2+z^2 = 1$ of a cavity in which a charge $q = 1$ is placed at distance r from the centre.

72. Calculate to the first order in ε the effect that the influence of the flattening of the earth ($\varepsilon \approx 1/300$) on the gravitational field of the earth has on the distance of the moon (assuming the earth to be homogeneous).

73. Find (to the first order in ε) the influence of the imperfection of an almost spherical capacitor $R = 1+\varepsilon f(\varphi, \theta)$ on its capacity.

74. Sketch the graph of $u(x, 1)$, if $0 \leqslant x \leqslant 1$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u\,|_{t=0} = x^2, \quad u\,|_{x^2=x} = x^2.$$

75. On account of the annual fluctuation of temperature the ground at the town of N freezes to a depth of 2 metres. To what depth would it freeze on account of a daily fluctuation of the same amplitude?

76. Investigate the behaviour at $t \to +\infty$ of the solution of the problem

$$u_t + (u \sin x)_x = \varepsilon u_{xx}, \quad u\,|_{t=0} \equiv 1, \quad \varepsilon \ll 1.$$

77. Find the eigenvalues and their multiplicities of the Laplace operator $\Delta = \operatorname{div}\operatorname{grad}$ on a sphere of radius R in Euclidean space of dimension n.

78. Solve the Cauchy problem

$$\frac{\partial^2 A}{\partial t^2} = 9\frac{\partial^2 A}{\partial x^2} - 2B, \quad \frac{\partial^2 B}{\partial t^2} = 6\,\frac{\partial^2 B}{\partial x^2} - 2A,$$

$$A\,|_{t=0} = \cos x, \quad B\,|_{t=0} = 0, \quad \left.\frac{\partial A}{\partial t}\right|_{t=0} = \left.\frac{\partial B}{\partial t}\right|_{t=0} = 0.$$

79. How many solutions has the boundary-value problem

$$u_{xx} + \lambda u = \sin x, \quad u\,(0) = u\,(\pi) = 0?$$

80. Solve the equation

$$\int_0^1 (x+y)^2 u(x)\, dx = \lambda u(y) + 1.$$

81. Find the Green's function of the operator $d^2/dx^2 - 1$ and solve the equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy = e^{-x^2}.$$

82. For what values of the velocity c does the equation $u_t = u - u^2 + u_{xx}$ have a solution in the form of a travelling wave $u = \varphi(x-ct)$, $\varphi(-\infty) = 1$, $\varphi(\infty) = 0$, $0 \leqslant u \leqslant 1$?

83. Find solutions of the equation $u_t = u_{xxx} + uu_x$ in the form of a travelling wave $u = \varphi(x-ct)$, $\varphi(\pm\infty) = 0$.

84. Find the number of positive and negative squares in the canonical form of the quadratic form $\sum_{i<j} (x_i - x_j)^2$ in n variables. The same for the form $\sum_{i<j} x_i x_j$.

85. Find the lengths of the principal axes of the ellipsoid

$$\sum_{i \leqslant j} x_i x_j = 1.$$

86. Through the centre of a cube (tetrahedron, icosahedron) draw a straight line in such a way that the sum of the squares of its distances from the vertices is a) minimal, b) maximal.

87. Find the derivatives of the lengths of the semiaxes of the ellipsoid $x^2+y^2+z^2+xy+yz+zx = 1+\varepsilon xy$ with respect to ε at $\varepsilon = 0$.

88. How many figures can be obtained by intersecting the infinite-dimensional cube $|x_k| \leqslant 1$, $k = 1, 2, \ldots$, with a two-dimensional plane?

89. Calculate the sum of vector products $[[x, y], z] + [[y, z], x] + [[z, x], y]$.

90. Calculate the sum of matrix commutators $[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$, where $[A, B] = AB - BA$.

91. Find the Jordan normal form of the operator $e^{d/dt}$ in the space of quasi-polynomials $\{e^{\lambda t} p(t)\}$, where the degree of the polynomial p is less than 5, and of the operator ad_A, $B \mapsto [A, B]$, in the space of $n \times n$ matrices B, where A is a diagonal matrix.

92. Find the orders of the subgroups of the group of rotations of the cube, and find its normal subgroups.

93. Decompose the space of functions defined on the vertices of a cube into invariant subspaces irreducible with respect to the group of a) its symmetries, b) its rotations.

94. Decompose a 5-dimensional real linear space into the irreducible invariant subspaces of the group generated by cyclic permutations of the basis vectors.

95. Decompose the space of homogeneous polynomials of degree 5 in (x, y, z) into irreducible subspaces invariant with respect to the rotation group $SO(3)$.

96. Each of 3600 subscribers of a telephone exchange calls it once an hour on average. What is the probability that in a given second 5 or more calls are received? Estimate the mean interval of time between such seconds $(i, i+1)$.

97. A particle performing a random walk on the integer points of the semi-axis $x \geqslant 0$ moves a distance 1 to the right with probability a, and to the left with probability b, and stands still in the remaining cases (if $x = 0$, it stands still instead of moving to the left). Determine the steady-state probability distribution, and also the expectation of x and x^2 over a long time, if the particle starts at the point 0.

98. In the game of "Fingers", N players stand in a circle and simultaneously thrust out their right hands, each with a certain number of fingers showing. The total number of fingers shown is counted out round the circle from the leader, and the player on whom the count stops is the winner. How large must N be for a suitably chosen group of $N/10$ players to contain a winner with probability at least 0.9? How does the probability that the leader wins behave as $N \to \infty$?

99. One player conceals a 10 or 20 copeck coin, and the other guesses its value. If he is right he gets the coin, if wrong he pays 15 copecks. Is this a fair game? What are the optimal mixed strategies for both players?

100. Find the mathematical expectation of the area of the projection of a cube with edge of length 1 onto a plane with an isotropically distributed random direction of projection.

Translated by C.J. Shaddock