Condensed Matter - HW 5 :: Zero Sound Attenuation

PHSX 545

Problem 1

Write the transport equation with collision term in τ approximation:

$$(\omega - \mathbf{q}\mathbf{v}_f)\nu_{\hat{\mathbf{p}}} - \mathbf{q}\mathbf{v}_f \int \frac{d\Omega_{\hat{p}'}}{4\pi} F^s(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'} - \mathbf{q}\mathbf{v}_f\, U = -\frac{i}{\tau}[\nu_{\hat{\mathbf{p}}} - \nu_0 - \nu_1 P_1(\mathbf{q} \cdot \hat{\mathbf{p}})]$$

where we subtracted $\ell = 0$ and $\ell = 1$ terms in collision integral to satisfy the particle and momentum conservation laws.

(a) By projecting out different $P_\ell(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})$ harmonics derive general equation for ν_ℓ amplitudes directly from this equation, without dividing by $(\omega - \mathbf{q}\mathbf{v}_f)$ throughout (the latter we did in class, which resulted in $\Omega_{\ell\ell'}(s)$ functions). Hint: use the product property and one of the recursion relations ($xP_n(x) = \dots$) of Legendre polynomials.

(b) Assume $F^s(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ has non-zero $F^s_{\ell=0,1,2}$ terms only, and drop all others, $F^s_{\ell>2} = 0$. Write down equations for $\ell = 0, 1, 2, 3$ explicitly. Show that the $\ell = 0$ equation corresponds to particle number conservation, and try to show that $\ell = 1$ equation is momentum conservation (you might want to recall assignment two weeks ago).

(c) In the large $s = \omega/qv_f$ limit show that you can terminate the ν_ℓ series at $\ell = 2$. Set components $\ell > 2$ to zero and use equations for first three components ($\nu_{\ell=0,1,2}$) to find dispersion relation for sound wave s.

(d) Investigate the transition from first ($\omega\tau \ll 1$, expansion in $\omega\tau$) to zero ($\omega\tau \gg 1$, expansion in $1/\omega\tau$) sound, and explicitly determine temperature dependence of attenuation ($q = q' + iq''$) in the two limits.

Answer of exercise 1

(a) Starting from

$$(\omega - \mathbf{q}\mathbf{v}_f)\nu_{\hat{\mathbf{p}}} - \mathbf{q}\mathbf{v}_f \int \frac{d\Omega_{\hat{p}'}}{4\pi} F^s(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'} - \mathbf{q}\mathbf{v}_f\, U = -\frac{i}{\tau}[\nu_{\hat{\mathbf{p}}} - \nu_0 - \nu_1 P_1(\mathbf{q}\cdot\hat{\mathbf{p}})]$$

we expand in Legendre polynomials the distribution function and the interaction parameters

$$\nu_{\hat{\mathbf{p}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) = \sum_\ell \nu_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) \qquad F^s(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) = \sum_\ell F^s_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})$$

and use the product property of the Legendre polynomials

$$\int \frac{d\Omega_{\hat{p}}}{4\pi} P_{\ell_1}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_1)P_{\ell_2}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_2) = \frac{\delta_{\ell_1\ell_2}}{2\ell_1+1}P_{\ell_1}(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)$$

to write the second term as a sum:

$$(\omega - qv_f\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\sum_\ell \nu_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) - qv_f\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\sum_\ell \frac{F^s_\ell}{2\ell+1}\nu_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) - v_f q\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\, U = -\frac{i}{\tau}\sum_{\ell=2}\nu_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})$$

Then use the recursion relation to express product $(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})$ as a sum of Legendre polynomilas of $\ell \pm 1$ order:

$$xP_\ell(x) = \frac{\ell}{2\ell+1}P_{\ell-1}(x) + \frac{\ell+1}{2\ell+1}P_{\ell+1}(x)$$

For example, several first recursion relations for the polynomials are:

$$xP_0(x) = P_1(x) \qquad xP_1(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x) \qquad xP_2(x) = \frac{2}{5}P_1(x) + \frac{3}{5}P_3(x)$$

We use to write the ℓ-sums as

$$(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\sum_\ell P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})A_\ell = \sum_\ell\left(\frac{\ell}{2\ell+1}P_{\ell-1}(x) + \frac{\ell+1}{2\ell+1}P_{\ell+1}(x)\right)A_\ell = \sum_\ell \frac{\ell+1}{2\ell+3}P_\ell(x)A_{\ell+1} + \sum_\ell \frac{\ell}{2\ell-1}P_\ell(x)A_{\ell-1}$$

Since different harmonics are orthogonal we can read off the equations for various ℓ's:

$$\omega\nu_\ell - qv_f\left(\frac{\ell}{2\ell-1}\nu_{\ell-1} + \frac{\ell+1}{2\ell+3}\nu_{\ell+1}\right) - qv_f\left(\frac{F^s_{\ell-1}\ell}{(2\ell-1)^2}\nu_{\ell-1} + \frac{F^s_{\ell+1}(\ell+1)}{(2\ell+3)^2}\nu_{\ell+1}\right) - qv_f\delta_{\ell 1}U = -\frac{i}{\tau}\begin{cases} 0\ ,\ \ell=0 \\ 0\ ,\ \ell=1 \\ \nu_\ell\ ,\ \ell=2,3,\ldots \end{cases} \tag{1}$$

(b) which for $\ell = 0, 1, 2, 3$ give

$$\ell = 0 \qquad \omega\nu_0 - qv_f\frac{1}{3}\nu_1 - qv_f\frac{F_1^s}{3^2}\nu_1 = \omega\nu_0 - qv_f\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 = 0 \tag{2}$$

$$\ell = 1 \qquad \omega\nu_1 - qv_f(1 + F_0^s)\nu_0 - qv_f\frac{2}{5}\left(1 + \frac{F_2^s}{5}\right)\nu_2 = qv_f U \tag{3}$$

$$\ell = 2 \qquad \omega\nu_2 - qv_f\frac{2}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 - qv_f\frac{3}{7}\left(1 + \frac{F_3^s = 0}{7}\right)\nu_3 = -\frac{i}{\tau}\nu_2 \tag{4}$$

$$\ell = 3 \qquad \omega\nu_3 - qv_f\frac{3}{5}\left(1 + \frac{F_2^s}{3}\right)\nu_2 - qv_f\frac{4}{9}\left(1 + \frac{F_4^s = 0}{9}\right)\nu_4 = -\frac{i}{\tau}\nu_3 \tag{5}$$

To see that the first two equations correspond to the conservation laws we recall definitions of the particle density, current, and momentum tensor, and use $\delta n_{\mathbf{p}} = -\frac{\partial n^0_{\mathbf{p}}}{\partial \varepsilon_{\mathbf{p}}}\nu_{\hat{\mathbf{p}}}$ to obtain results familiar from one of the previous homework assignments:

$$\delta n = 2\sum_{\mathbf{p}} \delta n_{\mathbf{p}} = N_0 \int \frac{d\Omega_{\hat{p}}}{4\pi}\nu_{\hat{\mathbf{p}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) = N_0\nu_0$$

$$\delta\mathbf{j} = 2\sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}}\delta\bar{n}_{\mathbf{p}} = v_f N_0 \int \frac{d\Omega_{\hat{p}}}{4\pi}\hat{\mathbf{p}}\left(\nu_{\hat{\mathbf{p}}} + \int \frac{d\Omega_{\hat{p}'}}{4\pi}F^s(\hat{\mathbf{p}}\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'}\right) = \hat{\mathbf{q}}N_0 v_f\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1$$

$$\delta\mathbf{g} = m\delta\mathbf{j} = \hat{\mathbf{q}}N_0 p_f\frac{m}{m^*}\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 = \hat{\mathbf{q}}\frac{1}{3}N_0 p_f\nu_1$$

$$\Pi_{ij} = p_f v_f 2\sum_{\mathbf{p}}\hat{p}_i\hat{p}_j\delta\bar{n}_{\mathbf{p}} = p_f v_f N_0\int\frac{d\Omega_{\hat{p}}}{4\pi}\hat{p}_i\hat{p}_j\left(\nu_{\hat{\mathbf{p}}} + \int\frac{d\Omega_{\hat{p}'}}{4\pi}F^s(\hat{\mathbf{p}}\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'}\right)$$

$$= p_f v_f N_0\int\frac{d\Omega_{\hat{p}}}{4\pi}\hat{p}_i\hat{p}_j\left(\nu_0 + \nu_2 P_2 + F_0^s\nu_0 + \frac{1}{5}F_2^s\nu_2 P_2\right)$$

$$= N_0 p_f v_f\frac{1}{3}\delta_{ij}\left(1 + F_0^s\right)\nu_0 + N_0 p_f v_f\nu_2\left(1 + \frac{F_2^s}{5}\right)\frac{1}{15}\left(3\hat{q}_i\hat{q}_j - \delta_{ij}\right)$$

The conservation of particle number gives the $\ell = 0$ equation:

$$\frac{\partial}{\partial t}\delta n + \boldsymbol{\nabla}\cdot\delta\mathbf{j} \propto (\omega\delta n - \mathbf{q}\cdot\delta\mathbf{j}) = 0 \qquad \Rightarrow \qquad \omega N_0\nu_0 - N_0 qv_f\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 = 0$$

The momentum conservation equation includes $n_0 = \frac{2}{3}N_0\varepsilon_f = \frac{1}{3}N_0 v_f p_f$,

$$\frac{\partial}{\partial t}\delta g_i + \boldsymbol{\nabla}_j\Pi_{ij} + n_0\boldsymbol{\nabla}_i U \propto \omega\delta g_i - q_j\Pi_{ij} - q_i n_0 U = 0$$

$$\Rightarrow \qquad \hat{q}_i\frac{1}{3}N_0 p_f\;\nu_1\omega - \hat{q}_i\frac{1}{3}N_0 p_f v_f\left[q(1 + F_0^s)\nu_0 + \nu_2\left(1 + \frac{F_2^s}{5}\right)\frac{2}{5}q\right] = \hat{q}_i\frac{1}{3}N_0 p_f\;v_f qU$$

which after cancellation of common prefactor $\hat{q}_i\frac{1}{3}N_0 p_f$ does give $\ell = 1$ equation.

(c) We introduce variable

$$s = \frac{\omega}{q v_f}$$

If we divide equation (5) by ω we see that the ν_3 amplitude is $\sim \nu_2/s$ - small in $s \gg 1$ limit, and all consequitive amplitudes small too. We neglect them. We can say the same about $\nu_2/\nu_1 \sim 1/s$, but ν_2 equation is the first one that contains the scattering time, so we want to keep it.

Keeping only $\nu_{0,1,2}$ amplitudes we re-write $\ell = 0, 1, 2$ equations as

$$\begin{aligned} s\nu_0 - \frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 &= 0 \\ s\nu_1 - (1 + F_0^s)\,\nu_0 - \frac{2}{5}\left(1 + \frac{F_2^s}{5}\right)\nu_2 &= U \\ \nu_2 - \frac{2}{s}\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)\nu_1 &= -\frac{i}{\omega\tau}\nu_2 \end{aligned} \tag{6}$$

Or as a matrix:

$$\begin{pmatrix} s & -\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right) & 0 \\ -(1 + F_0^s) & s & -\frac{2}{5}\left(1 + \frac{F_2^s}{5}\right) \\ 0 & -\frac{2}{s}\frac{1}{3}\left(1 + \frac{F_1^s}{3}\right) & 1 + \frac{i}{\omega\tau} \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ U \\ 0 \end{pmatrix} \tag{7}$$

The sound dispersion equation is given by the condition of zero determinant:

$$s^2\left(1 + \frac{i}{\omega\tau}\right) - \frac{1}{3}\left(1 + \frac{F_1^s}{3}\right)(1 + F_0^s)\left(1 + \frac{i}{\omega\tau}\right) - \frac{4}{15}\left(1 + \frac{F_1^s}{3}\right)\left(1 + \frac{F_2^s}{5}\right) = 0 \tag{8}$$

(d) The first sound exists in the limit $\omega\tau \ll 1$ and we can write its dispersion as

$$\text{First sound:} \qquad s^2 = \frac{1}{3}\left(1+\frac{F_1^s}{3}\right)(1+F_0^s) - i(\omega\tau)\frac{4}{15}\left(1+\frac{F_1^s}{3}\right)\left(1+\frac{F_2^s}{5}\right) \tag{9}$$

where we kept only first order $\omega\tau$ term. From this relation the speed of first sound is

$$c_1^2 = \frac{v_f^2}{3}\left(1+\frac{F_1^s}{3}\right)(1+F_0^s)$$

and we can write the wave vector at a given frequency from relation:

$$\left(\frac{\omega}{c_1 q}\right)^2 = 1 - i(\omega\tau)\frac{4(1+F_2^s/5)}{5(1+F_0^s)} \qquad \Rightarrow \qquad q = \frac{\omega}{c_1}\left(1 - i(\omega\tau)\frac{4(1+F_2^s/5)}{5(1+F_0^s)}\right)^{-1/2} \approx \frac{\omega}{c_1} + i\frac{\omega^2\tau}{c_1}\frac{2(1+F_2^s/5)}{5(1+F_0^s)}$$

and the attenuation of the first sound is proportional to the scattering time $\tau \propto 1/T^2$:

$$q_1'' \sim \frac{\omega^2\tau(T)}{c_1} \propto \frac{\omega^2}{T^2}$$

The zero sound is in the limit $\omega\tau \gg 1$ and keeping only first-order terms in $1/\omega\tau$ we have

$$\text{Zero sound:} \qquad s^2 = \frac{1}{3}\left(1+\frac{F_1^s}{3}\right)(1+F_0^s) + \frac{4}{15}\left(1+\frac{F_1^s}{3}\right)\left(1+\frac{F_2^s}{5}\right)\left(1-\frac{i}{\omega\tau}\right) \tag{10}$$

The speed of zero sound is

$$c_0^2 = v_f^2\frac{1}{3}\left(1+\frac{F_1^s}{3}\right)(1+F_0^s) + v_f^2\frac{4}{15}\left(1+\frac{F_1^s}{3}\right)\left(1+\frac{F_2^s}{5}\right) \qquad \Rightarrow \qquad \frac{c_0^2-c_1^2}{c_1^2} = \frac{4}{5}\frac{1+F_2^s/5}{1+F_0^s}$$

and the dispersion and attenuation wave vector

$$\left(\frac{\omega}{c_0 q}\right)^2 = 1 - i\frac{1}{\omega\tau}\frac{(4/5)(1+F_2^s/5)}{(1+F_0^s)+(4/5)(1+F_2^s/5)} \qquad \Rightarrow \qquad q = \frac{\omega}{c_0}\left(1 - i\frac{1}{\omega\tau}\frac{(4/5)(1+F_2^s/5)}{(1+F_0^s)+(4/5)(1+F_2^s/5)}\right)^{-1/2}$$

with attenuation proportional to inverse scattering time and frequency independent!

$$q_0'' \sim \frac{1}{c_0\tau} \propto T^2$$

The general expression for sound mode that span both limits is

$$\left(\frac{\omega}{c_1 q}\right)^2 = 1 + \frac{\omega\tau}{i+\omega\tau}\frac{4(1+F_2^s/5)}{5(1+F_0^s)} = 1 + \frac{(\omega\tau)(\omega\tau - i)}{1+(\omega\tau)^2}\frac{4(1+F_2^s/5)}{5(1+F_0^s)}$$

and the wave vector is

$$q = q' + iq'' = \frac{\omega}{c_1}\left(1 + \frac{(\omega\tau)(\omega\tau-i)}{1+(\omega\tau)^2}\frac{4(1+F_2^s/5)}{5(1+F_0^s)}\right)^{-1/2}$$

with the attenuation being the imaginary part of this square root.