Problem 1

Write the transport equation with collision term in $\tau$ approximation:

$$
(\omega - qV_f)\nu_\hat{p} - qV_f \int \frac{d\Omega_{\hat{p}'}}{4\pi} F^s(\hat{p} \cdot \hat{p}') \nu_{p'} - qV_f U = -\frac{i}{\tau} [\nu_\hat{p} - \nu_0 - \nu_1 P_1(q \cdot \hat{p})]
$$

where we subtracted $\ell = 0$ and $\ell = 1$ terms in collision integral to satisfy the particle and momentum conservation laws.

(a) By projecting out different $P_\ell(\hat{p} \cdot \hat{q})$ harmonics derive general equation for $\nu_\ell$ amplitudes directly from this equation, without dividing by $(\omega - qV_f)$ throughout (the latter we did in class, which resulted in $\Omega_{\ell\ell}'(s)$ functions). Hint: use the product property and one of the recursion relations ($xP_n(x) = \ldots$) of Legendre polynomials.

(b) Assume $F^s(\hat{p} \cdot \hat{p}')$ has non-zero $F^s_{\ell=0,1,2}$ terms only, and drop all others, $F^s_{\ell>2} = 0$. Write down equations for $\ell = 0, 1, 2, 3$ explicitly. Show that the $\ell = 0$ equation corresponds to particle number conservation, and try to show that $\ell = 1$ equation is momentum conservation (you might want to recall assignment two weeks ago).

(c) In the large $s = \omega/qV_f$ limit show that you can terminate the $\nu_\ell$ series at $\ell = 2$. Set components $\ell > 2$ to zero and use equations for first three components ($\nu_{\ell=0,1,2}$) to find dispersion relation for sound wave $s$.

(d) Investigate the transition from first ($\omega \tau \ll 1$, expansion in $\omega \tau$) to zero ($\omega \tau \gg 1$, expansion in $1/\omega \tau$) sound, and explicitly determine temperature dependence of attenuation ($q = q' + iq''$) in the two limits.
Answer of exercise 1

(a) Starting from

\[ (\omega - q v_f) \nu_{\hat{p}} - q v_f \int \frac{d\Omega_{\hat{p}}}{4\pi} F^s(\hat{p} \cdot \hat{p}') \nu_{\hat{p}'} - q v_f U = -\frac{i}{\tau} [\nu_{\hat{p}} - \nu_0 - \nu_1 P_1(q \cdot \hat{p})] \]

we expand in Legendre polynomials the distribution function and the interaction parameters

\[ \nu_{\hat{p}}(\hat{p} \cdot \hat{q}) = \sum_\ell \nu_\ell P_\ell(\hat{p} \cdot \hat{q}) \quad F^s(\hat{p} \cdot \hat{q}) = \sum_\ell F^s_\ell P_\ell(\hat{p} \cdot \hat{q}) \]

and use the product property of the Legendre polynomials

\[ \int \frac{d\Omega_{\hat{p}}}{4\pi} P_{\ell_1}(\hat{p} \cdot \hat{k}_1) P_{\ell_2}(\hat{p} \cdot \hat{k}_2) = \frac{\delta_{\ell_1, \ell_2}}{2\ell_1 + 1} P_{\ell_1}(\hat{k}_1 \cdot \hat{k}_2) \]

to write the second term as a sum:

\[ (\omega - q v_f \hat{p} \cdot \hat{q}) \sum_\ell \nu_\ell P_\ell(\hat{p} \cdot \hat{q}) - q v_f \hat{p} \cdot \hat{q} \sum_\ell \frac{F^s_\ell}{2\ell + 1} \nu_\ell P_\ell(\hat{p} \cdot \hat{q}) - v_f q \hat{p} \cdot \hat{q} U = -\frac{i}{\tau} \sum_\ell \nu_\ell P_\ell(\hat{p} \cdot \hat{q}) \]

Then use the recursion relation to express product \((\hat{p} \cdot \hat{q}) P_\ell(\hat{p} \cdot \hat{q})\) as a sum of Legendre polynomials of \(\ell \pm 1\) order:

\[ xP_\ell(x) = \frac{\ell}{2\ell + 1} P_{\ell-1}(x) + \frac{\ell + 1}{2\ell + 1} P_{\ell+1}(x) \]

For example, several first recursion relations for the polynomials are:

\[ xP_0(x) = P_1(x) \quad xP_1(x) = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \quad xP_2(x) = \frac{2}{5} P_1(x) + \frac{3}{5} P_3(x) \]

We use to write the \(\ell\)-sums as

\[ (\hat{p} \cdot \hat{q}) \sum_\ell P_\ell(\hat{p} \cdot \hat{q}) A_\ell = \sum_\ell \left( \frac{\ell}{2\ell + 1} P_{\ell-1}(x) + \frac{\ell + 1}{2\ell + 1} P_{\ell+1}(x) \right) A_\ell = \sum_\ell \frac{\ell + 1}{2\ell + 3} P_\ell(x) A_{\ell+1} + \sum_\ell \frac{\ell}{2\ell - 1} P_\ell(x) A_{\ell-1} \]

Since different harmonics are orthogonal we can read off the equations for various \(\ell\)’s:

\[
\omega \nu_\ell - q v_f \left( \frac{\ell}{2\ell - 1} \nu_{\ell-1} + \frac{\ell + 1}{2\ell + 3} \nu_{\ell+1} \right) - q v_f \left( \frac{F^s_{\ell-1}(\ell)}{(2\ell - 1)^2} \nu_{\ell-1} + \frac{F^s_{\ell+1}(\ell + 1)}{(2\ell + 3)^2} \nu_{\ell+1} \right) - q v_f \delta_{\ell 1} U = -\frac{i}{\tau} \left\{ \begin{array}{ll}
0 & , \ell = 0 \\
0 & , \ell = 1 \\
\nu_\ell & , \ell = 2, 3, \ldots
\end{array} \right\}
\]

(1)
(b) which for $\ell = 0, 1, 2, 3$ give

\begin{align*}
\ell = 0 & \quad \omega \nu_0 - q v_f \frac{1}{3} \nu_1 - q v_f \frac{F_s^1}{3^2} \nu_1 = \omega \nu_0 - q v_f \left(1 + \frac{F_s^1}{3}\right) \nu_1 = 0 \\
\ell = 1 & \quad \omega \nu_1 - q v_f (1 + F_0^s) \nu_0 - q v_f \frac{2}{5} \left(1 + \frac{F_2^s}{5}\right) \nu_2 = q v_f U \\
\ell = 2 & \quad \omega \nu_2 - q v_f \frac{2}{3} \left(1 + \frac{F_1^s}{3}\right) \nu_1 - q v_f \frac{3}{7} \left(1 + \frac{F_3^s}{7}\right) \nu_3 = -\frac{i}{\tau} \nu_2 \\
\ell = 3 & \quad \omega \nu_3 - q v_f \frac{3}{5} \left(1 + \frac{F_2^s}{5}\right) \nu_2 - q v_f \frac{4}{9} \left(1 + \frac{F_3^s}{9}\right) \nu_4 = -\frac{i}{\tau} \nu_3
\end{align*}

To see that the first two equations correspond to the conservation laws we recall definitions of the particle density, current, and momentum tensor, and use $\delta n_p = -\frac{\partial n_p}{\partial \epsilon} \nu_p$ to obtain results familiar from one of the previous homework assignments:

\[ \delta n = 2 \sum_p \delta n_p = N_0 \int \frac{d\Omega_p}{4\pi} \nu_p (\delta n_p) = N_0 \nu_0 \]

\[ \delta j = 2 \sum_p v_p \delta n_p = q f N_0 \int \frac{d\Omega_p}{4\pi} \hat{p} \left( \nu_p + \int \frac{d\Omega_{p'}}{4\pi} F^s (\hat{p} \hat{p'}) \nu_{p'} \right) = q f N_0 \frac{1}{3} \left(1 + \frac{F^s_1}{3}\right) \nu_1 \]

\[ \delta g = m \delta j = q f N_0 p f \frac{m}{m^*} \frac{1}{3} \left(1 + \frac{F^s_1}{3}\right) \nu_1 = q f N_0 p_f \nu_1 \]

\[ \Pi_{ij} = \frac{f f}{2} \sum_p \hat{p}_i \hat{p}_j \delta n_p = p f v_f N_0 \int \frac{d\Omega_p}{4\pi} \hat{p}_i \hat{p}_j \left( \nu_p + \int \frac{d\Omega_{p'}}{4\pi} F^s (\hat{p} \hat{p'}) \nu_{p'} \right) \]

\[ = p f v_f N_0 \int \frac{d\Omega_p}{4\pi} \hat{p}_i \hat{p}_j \left( \nu_0 + \nu_2 P_2 + F_0^s \nu_0 + \frac{1}{5} F_2^s \nu_2 P_2 \right) \]

\[ = N_0 p_f v_f \frac{1}{3} \nu_1 \left(1 + F_0^s\right) \nu_0 + N_0 p_f v_f \nu_2 \left(1 + \frac{F_2^s}{5}\right) \frac{1}{15} \left(3 \delta_i \delta_j - \delta_{ij}\right) \]

The conservation of particle number gives the $\ell = 0$ equation:

\[ \frac{\partial}{\partial t} \delta n + \nabla \cdot \delta j \propto (\omega \delta n - q \cdot \delta j) = 0 \quad \Rightarrow \quad \omega N_0 \nu_0 - N_0 q v_f \frac{1}{3} \left(1 + \frac{F^s_1}{3}\right) \nu_1 = 0 \]

The momentum conservation equation includes $n_0 = \frac{2}{3} N_0 \varepsilon_f = \frac{1}{3} N_0 v_f p_f$.

\[ \frac{\partial}{\partial t} \delta g_i + \nabla_j \Pi_{ij} + n_0 \nabla_i U \propto \omega \delta g_i - q_j \Pi_{ij} - q_i n_0 U = 0 \]

\[ \Rightarrow \quad \hat{q}_i \frac{1}{3} N_0 p_f \nu_1 \omega - \hat{q}_i \frac{1}{3} N_0 p_f v_f \left[ q (1 + F_0^s) \nu_0 + \nu_2 \left(1 + \frac{F_2^s}{5}\right) \frac{2}{5} q \right] = \hat{q}_i \frac{1}{3} N_0 p_f \nu f q U \]

which after cancellation of common prefactor $\hat{q}_i \frac{1}{3} N_0 p_f$ does give $\ell = 1$ equation.
(c) We introduce variable

\[ s = \frac{\omega}{qv_f} \]

If we divide equation (5) by \( \omega \) we see that the \( \nu_3 \) amplitude is \( \sim \nu_2/s \) - small in \( s \gg 1 \) limit, and all consequitive amplitudes small too. We neglect them. We can say the same about \( \nu_2/\nu_1 \sim 1/s \), but \( \nu_2 \) equation is the first one that contains the scattering time, so we want to keep it.

Keeping only \( \nu_{0,1,2} \) amplitudes we re-write \( \ell = 0, 1, 2 \) equations as

\[
\begin{align*}
sv_0 - \frac{1}{3} \left(1 + \frac{F_s^1}{3}\right) v_1 &= 0 \\
sv_1 - (1 + F_s^0) v_0 - \frac{2}{5} \left(1 + \frac{F_s^2}{5}\right) v_2 &= U \\
v_2 - \frac{2}{s} \frac{4}{3} \left(1 + \frac{F_s^1}{3}\right) v_1 &= -\frac{i}{\omega \tau} v_2
\end{align*}
\]  

(6)

Or as a matrix:

\[
\begin{pmatrix}
 s & -\frac{1}{3} \left(1 + \frac{F_s^1}{3}\right) & 0 \\
-(1 + F_s^0) & s & -\frac{2}{5} \left(1 + \frac{F_s^2}{5}\right) \\
0 & -\frac{2}{s} \frac{4}{3} \left(1 + \frac{F_s^1}{3}\right) & 1 + \frac{i}{\omega \tau}
\end{pmatrix}
\begin{pmatrix}
\nu_0 \\
\nu_1 \\
\nu_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
U \\
0
\end{pmatrix}
\]  

(7)

The sound dispersion equation is given by the condition of zero determinant:

\[
s^2 \left(1 + \frac{i}{\omega \tau}\right) - \frac{1}{3} \left(1 + \frac{F_s^1}{3}\right) \left(1 + \frac{F_s^0}{3}\right) \left(1 + \frac{i}{\omega \tau}\right) - \frac{4}{15} \left(1 + \frac{F_s^1}{3}\right) \left(1 + \frac{F_s^2}{5}\right) = 0
\]  

(8)
(d) The first sound exists in the limit \( \omega \tau \ll 1 \) and we can write its dispersion as

\[
\text{First sound: } s^2 = \frac{1}{3} \left( 1 + \frac{F_s^1}{3} \right) (1 + F_0^s) - i(\omega \tau) \frac{4}{15} \left( 1 + \frac{F_s^1}{3} \right) \left( 1 + \frac{F_s^5}{5} \right)
\]

where we kept only first order \( \omega \tau \) term. From this relation the speed of first sound is

\[
c_1^2 = \frac{v_f^2}{3} \left( 1 + \frac{F_s^5}{3} \right) (1 + F_0^s)
\]

and we can write the wave vector at a given frequency from relation:

\[
\left( \frac{\omega}{c_1 q} \right)^2 = 1 - i(\omega \tau) \frac{4(1 + F_s^2/5)}{5(1 + F_0^s)} \Rightarrow q = \frac{\omega}{c_1} \left( 1 - i(\omega \tau) \frac{4(1 + F_s^2/5)}{5(1 + F_0^s)} \right)^{-1/2} \approx \frac{\omega}{c_1} + \frac{\omega^2 \tau}{c_1} \frac{2(1 + F_s^2/5)}{5(1 + F_0^s)}
\]

and the attenuation of the first sound is proportional to the scattering time \( \tau \propto 1/T^2 \):

\[
q'' = \frac{\omega^2 \tau(T)}{c_1} \propto \frac{\omega^2}{T^2}
\]

The zero sound is in the limit \( \omega \tau \gg 1 \) and keeping only first-order terms in \( 1/\omega \tau \) we have

\[
\text{Zero sound: } s^2 = \frac{1}{3} \left( 1 + \frac{F_s^1}{3} \right) (1 + F_0^s) + \frac{4}{15} \left( 1 + \frac{F_s^1}{3} \right) \left( 1 + \frac{F_s^5}{5} \right) \left( 1 - \frac{i}{\omega \tau} \right)
\]

The speed of zero sound is

\[
c_0^2 = v_f^2 \frac{1}{3} \left( 1 + \frac{F_s^5}{3} \right) (1 + F_0^s) + v_f^2 \frac{4}{15} \left( 1 + \frac{F_s^5}{3} \right) \left( 1 + \frac{F_s^5}{5} \right) \Rightarrow \frac{c_0^2 - c_1^2}{c_1^2} = \frac{4}{5} \frac{1 + F_s^2/5}{1 + F_0^s}
\]

and the dispersion and attenuation wave vector

\[
\left( \frac{\omega}{c_0 q} \right)^2 = 1 - \frac{1}{\omega \tau} \frac{(4/5)(1 + F_s^2/5)}{(1 + F_0^s) + (4/5)(1 + F_s^2/5)} \Rightarrow q = \frac{\omega}{c_0} \left( 1 - \frac{1}{\omega \tau} \frac{(4/5)(1 + F_s^2/5)}{(1 + F_0^s) + (4/5)(1 + F_s^2/5)} \right)^{-1/2}
\]

with attenuation proportional to inverse scattering time and frequency independent!

\[
q'' \sim \frac{1}{c_0 \tau} \propto T^2
\]

The general expression for sound mode that span both limits is

\[
\left( \frac{\omega}{c_1 q} \right)^2 = 1 + \frac{\omega \tau}{i + \omega \tau} \frac{4(1 + F_s^2/5)}{5(1 + F_0^s)} = 1 + \frac{(\omega \tau)(\omega \tau - i) 4(1 + F_s^2/5)}{1 + (\omega \tau)^2} \frac{4(1 + F_s^2/5)}{5(1 + F_0^s)}
\]

and the wave vector is

\[
q = q' + iq'' = \frac{\omega}{c_1} \left( 1 + \frac{(\omega \tau)(\omega \tau - i) 4(1 + F_s^2/5)}{1 + (\omega \tau)^2} \frac{4(1 + F_s^2/5)}{5(1 + F_0^s)} \right)^{-1/2}
\]

with the attenuation being the imaginary part of this square root.