Condensed Matter - HW3 :: Fermi Liquid Currents

PHSX 545

Problem 1

Various currents in Fermi liquid theory are given by the following expressions (spin-independent):

$$\mathbf{j} = \frac{1}{V}\sum_{\mathbf{p}} (\boldsymbol{\nabla}_{\mathbf{p}} \varepsilon^0_{\mathbf{p}})\, \delta\bar{n}_{\mathbf{p}} \qquad \text{particle current}$$

$$\Pi_{ij} = \frac{1}{V}\sum_{\mathbf{p}} p_i \frac{\partial \varepsilon^0_{\mathbf{p}}}{\partial p_j} \delta\bar{n}_{\mathbf{p}} \qquad \text{momentum current}$$

$$\mathbf{q} = \frac{1}{V}\sum_{\mathbf{p}} \varepsilon^0_{\mathbf{p}} (\boldsymbol{\nabla}_{\mathbf{p}} \varepsilon^0_{\mathbf{p}})\, \delta\bar{n}_{\mathbf{p}} \qquad \text{energy current (heat)}$$

where $\varepsilon^0_{\mathbf{p}}$ is energy in global equilibrium, and $\bar{n}_{\mathbf{k}}$ is deviation of distribution function from *local equilibrium*, and that includes interactions between quasiparticles.

(a) Calculate the particle current for a *single* excitation at momentum $\mathbf{p}$. Hint: in the absence of quasiparticle interactions this would have been just the group velocity $\mathbf{v}_p \approx v_f \hat{\mathbf{p}}$ of the particle. With quasiparticle interactions it will be a different velocity $\mathbf{u}$, that beside $\mathbf{v}_{\mathbf{p}}$ includes *backflow* currents from all other quasiparticles disturbed by the motion of the original one. Is it consistent with Galilean invariance principle (remember that we define co-moving reference frame by condition $\mathbf{j}' = 0$ - no particle current, in this case $\mathbf{p}' = 0$)?

(b) Calculate momentum and energy currents. For momentum current use symmetry arguments to identify Fermi liquid parameters $F^{s,a}_\ell$ that enter Π_{ij}. Theoretically minded part of the class are strongly encouraged to do this calculation fully.

The following expressions might help:

$$\int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i = 0 \qquad \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j = \frac{1}{3}\delta_{ij} \qquad \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j \hat{k}_m = 0 \qquad \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j \hat{k}_m \hat{k}_n = \frac{1}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in})$$

Answer of exercise 1

(a) The single particle is added into state $\mathbf{p}$ so that

$$\delta n_{\mathbf{k}} = \delta(\mathbf{k} - \mathbf{p})$$

- this is change of the distribution function relative to the *global(!)* equilibrium. This particle perturbs other quasi-particles through interactions and we write

$$\mathbf{j_p} = \sum_{\mathbf{k}} \mathbf{v_k} \left(\delta n_{\mathbf{k}} - \frac{\partial n_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}} \delta\varepsilon_{\mathbf{k}} \right) \qquad \frac{\partial n_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}} \approx -\delta(\varepsilon_{\mathbf{k}} - \mu)$$

where (sum over spins performed)

$$\delta\varepsilon_{\mathbf{k}} = 2 \sum_{\mathbf{k}'} f^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \delta n_{\mathbf{k}'} = 2 f^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})$$

The integration over $\mathbf{k}$ now gives

$$\mathbf{j_p} \approx v_f \hat{\mathbf{p}} + N_f \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} (v_f \hat{\mathbf{k}})\, 2 f^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) = v_f \hat{\mathbf{p}} + 2N_f (v_f) \frac{f^s}{3} \hat{\mathbf{p}} = v_f \hat{\mathbf{p}} \left(1 + \frac{F_1^s}{3} \right)$$

This expression we can re-write using definition of the effective mass

$$\mathbf{j_p} = \frac{\mathbf{p}_f}{m^*} \left(1 + \frac{F_1^s}{3} \right) = \text{ Galilean invariant system } m^* = m(1 + F_1^s/3) \ = \frac{\mathbf{p}_f}{m}$$

where m is bare mass of the particle.

This is consistent with the Galilean invariance. Velocity $\mathbf{u} \equiv \mathbf{p}/m$ defines a reference frame in which there is no particle current:

$$\mathbf{j}' \propto \mathbf{p}' = \mathbf{p} - m\mathbf{u} = 0$$

since the mass that enters this expression is the bare mass!

NOTE:

We can understand this expression as follows. Say we have a particle in state $\mathbf{p}$, $\delta n_{\mathbf{p}\sigma} = 1$. This particle perturbs the energies of other states by shifting their energies and thus modifying the 'ground state' distribution:

$$\delta\varepsilon_{\mathbf{p}'\sigma'} = f_{\sigma'\sigma}(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \delta n_{\mathbf{p}\sigma} = f_{\sigma'\sigma}(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \qquad \delta n_{\mathbf{p}'\sigma'} = \frac{\partial n^0}{\partial \varepsilon_{\mathbf{p}'}} \delta\varepsilon_{\mathbf{p}'\sigma'} = \frac{\partial n^0}{\partial \varepsilon_{\mathbf{p}'}} f_{\sigma'\sigma}(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}})$$

This change in the distribution can be seen as the "backflow" current, and we have to calculate the current due to the presence of the original particle *relative* to the "backflow" which is the new 'ground state':

$$\mathbf{j}_{\hat{\mathbf{p}}} = \mathbf{v_p} - \text{'backflow'} = \mathbf{v_p} - \sum_{\sigma'\hat{\mathbf{p}}'} \mathbf{v}_{\mathbf{p}'} \delta n_{\mathbf{p}'\sigma'} = \mathbf{v_p} - \sum_{\sigma'\hat{\mathbf{p}}'} \mathbf{v}_{\mathbf{p}'} \frac{\partial n^0}{\partial \varepsilon_{\mathbf{p}'}} f_{\sigma'\sigma}(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) = \mathbf{v_p} - \sum_{\hat{\mathbf{p}}'} \mathbf{v}_{\mathbf{p}'} \frac{\partial n^0}{\partial \varepsilon_{\mathbf{p}'}}\, 2 f^s(\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}})$$

$$= \mathbf{v_p} + v_f \frac{F_1^s}{3} \hat{\mathbf{p}}$$

(b) The energy current is exactly similar to the particle current:

$$\mathbf{q} = \frac{1}{V} \sum_{\mathbf{k}} \varepsilon^0_{\mathbf{k}} (\boldsymbol{\nabla}_{\mathbf{k}} \varepsilon^0_{\mathbf{k}})\, \delta \bar{n}_{\mathbf{k}}$$

where we take

$$\varepsilon^0_{\mathbf{p}} = \xi_{\mathbf{p}} + \varepsilon_f = v_f(p - p_f) + \varepsilon_f \qquad \boldsymbol{\nabla}_{\mathbf{p}} \varepsilon^0_{\mathbf{p}} = v_f \hat{\mathbf{p}}$$

and so

$$\mathbf{q} = \sum_{\mathbf{k}} \varepsilon^0_{\mathbf{k}} v_f \hat{\mathbf{k}} \left(\delta n_{\mathbf{k}} - \frac{\partial n^0_{\mathbf{k}}}{\partial \varepsilon^0_{\mathbf{k}}} \sum_{\mathbf{k}'} 2 f^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \delta n_{\mathbf{k}'} \right)$$

with the derivative $\dfrac{\partial n_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}} = -\delta(\varepsilon^0_{\mathbf{k}} - \varepsilon_f)$. In the second term we do integration over $\mathbf{k}$ first,

$$\mathbf{q} = \sum_{\mathbf{k}} \varepsilon^0_{\mathbf{k}} v_f \hat{\mathbf{k}} \delta n_{\mathbf{k}} + \varepsilon_f v_f \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{\mathbf{k}} \sum_{\mathbf{k}'} F^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \delta n_{\mathbf{k}'}$$

After $\hat{\mathbf{k}}$ angle integration we replace $\mathbf{k}' \to \mathbf{k}$ and obtain:

$$\mathbf{q} = \sum_{\mathbf{k}} \varepsilon^0_{\mathbf{k}} v_f \hat{\mathbf{k}} \delta n_{\mathbf{k}} + \varepsilon_f \frac{F^s_1}{3} \sum_{\mathbf{k}} (v_f \hat{\mathbf{k}}) \delta n_{\mathbf{k}} = \sum_{\mathbf{k}} v_f \hat{\mathbf{k}} \left(\varepsilon^0_{\mathbf{k}} + \varepsilon_f \frac{F^s_1}{3} \right) \delta n_{\mathbf{k}}$$

So for the energy current we have

$$\mathbf{q}_{\mathbf{p}} = v_f \hat{\mathbf{p}} \left(\varepsilon^0_{\mathbf{p}} + \varepsilon_f \frac{F^s_1}{3} \right) = \xi^0_{\mathbf{p}} v_f \hat{\mathbf{p}} + \varepsilon_f \left(1 + \frac{F^s_1}{3} \right) v_f \hat{\mathbf{p}}$$

For the momentum flux we have

$$\Pi_{ij} = \frac{1}{V} \sum_{\mathbf{p}} p_i \frac{\partial \varepsilon^0_{\mathbf{p}}}{\partial p_j} \delta \bar{n}_{\mathbf{p}} = p_i v_f \hat{p}_j - \sum_{\mathbf{k}} k_i v_f \hat{k}_j \frac{\partial n_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}} \sum_{\mathbf{k}'} 2 f^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \delta n_{\mathbf{k}'}$$

$$= p \hat{p}_i v_f \hat{p}_j + p_f v_f \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j F^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})$$

From the symmetry, the last integral is non zero when $\hat{k}_i \hat{k}_j$ overlaps with one of the harmonics of $F^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})$. Since the product of two unit vectors can generally be expanded in spherical harmonics of $\ell = 0, 1, 2$, one expects that there will be contributions from interaction terms with $F^s_{0,1,2}$.

Explicitly we write for the interaction function

$$F^s(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) = F^s_0 + F^s_1 P_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) + F^s_2 P_2(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) = F^s_0 + F^s_1 (\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) + F^s_2 \frac{3(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})^2 - 1}{2}$$

Using the provided integral relations we can write for the quadrupolar term:

$$\int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j (\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})^2 = \hat{p}_m \hat{p}_n \int \frac{d\Omega_{\hat{k}}}{4\pi} \hat{k}_i \hat{k}_j \hat{k}_m \hat{k}_n = \frac{1}{15} (\delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{jm} \delta_{in}) \hat{p}_m \hat{p}_n = \frac{1}{15} \delta_{ij} + \frac{2}{15} \hat{p}_i \hat{p}_j$$

and for the momentum flux we have

$$\Pi_{ij} = p v_f \, \hat{p}_i \hat{p}_j + p_f v_f \left(F^s_0 \frac{1}{3} \delta_{ij} - \frac{F^s_2}{2} \frac{1}{3} \delta_{ij} + \frac{3 F^s_2}{2} \left[\frac{1}{15} \delta_{ij} + \frac{2}{15} \hat{p}_i \hat{p}_j \right] \right)$$

$$= v_f \left(p + p_f \frac{F^s_2}{5} \right) \hat{p}_i \hat{p}_j + p_f v_f \frac{1}{3} \left(F^s_0 - \frac{F^s_2}{5} \right) \delta_{ij}$$