Problem 1
(a) You don’t need to derive this part, and may simply consult the math references. Decompose Legendre’s polynomials in spherical harmonics
\[ P_\ell(\cos \theta) = P_\ell(\hat{p} \cdot \hat{p}') = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{p}) \]

(b) Using the above and orthogonality of spherical harmonics calculate
\[ \int \frac{d\Omega_{\hat{p}'}}{4\pi} P_\ell_1(\hat{p}_1 \cdot \hat{p}') P_\ell_2(\hat{p}_2 \cdot \hat{p}') = \cdots \]
where \( d\Omega_{\hat{p}'} = \sin \theta_{\hat{p}'} d\theta_{\hat{p}'} d\phi_{\hat{p}'} \) is the solid angle integration over directions of \( \hat{p}' \).
(c) Decompose \( f \) in Legendre polynomials and calculate:
\[ \int \frac{d\Omega_{\hat{p}'}}{4\pi} f(\hat{p} \cdot \hat{p}') \hat{p}' = \cdots \]

Problem 2
(a) Write down anticommutator and commutator of Pauli matrices using \( \delta_{ij} \) and \( \epsilon_{ijk} \) tensors
\[
[\sigma_i, \sigma_j]_+ = \cdots \\
[\sigma_i, \sigma_j]_- = \cdots
\]
(b) Express a product of Pauli matrices in terms of (a) results
\[ \sigma_i \sigma_j = \cdots \]
(c) Using (b) express in terms of scalar and vector products of \( a, b \)
\[ (a\sigma)(b\sigma) = \cdots \]
and find traces over spins
\[ \text{Tr} \{(a\sigma)(b\sigma)\} = \cdots \]
\[ \text{Tr} \{\sigma(a\sigma)(b\sigma)\} = \cdots \]
Answer of exercise 1  
(a) Decompose Legendre’s polynomials in spherical harmonics

\[ P_\ell(\cos \theta) = P_\ell(\hat{p} \cdot \hat{p}') = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell + 1} Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{p}') \]

where spherical harmonics defined with appropriate prefactors to give

\[ \int d\Omega \hat{p} Y_{\ell m}(\hat{p}) Y^*_{\ell' m'}(\hat{p}) = \delta_{\ell \ell'} \delta_{m m'} \]

(b) Using the above and orthogonality of spherical harmonics calculate

\[ \int \frac{d\Omega'}{4\pi} P_{\ell_1}(\hat{p}_1 \cdot \hat{p}') P_{\ell_2}(\hat{p}_2 \cdot \hat{p}') = \delta_{\ell_1 \ell_2} \frac{1}{2\ell_1 + 1} P_{\ell_1}(\hat{p}_1 \cdot \hat{p}_2) \]

(c) Decompose \( f \) in Legendre polynomials and calculate:

\[ \int \frac{d\Omega'}{4\pi} f(\hat{p} \cdot \hat{p}') \hat{p}' = \frac{f_p}{3} \hat{p} \]

The proof: multiply by a constant arbitrary vector \( \mathbf{u} \) and realize \( \hat{p}' \cdot \mathbf{u} = u P_1(\hat{p}' \cdot \hat{u}) \). Using previous item result

\[ \int \frac{d\Omega'}{4\pi} f(\hat{p} \cdot \hat{p}') \ u P_1(\hat{p}' \cdot \hat{u}) = \frac{f_p}{3} \ u P_1(\hat{p} \cdot \hat{u}) = \frac{f_p}{3} \mathbf{p} \cdot \mathbf{u} \]

and eliminate the \( \mathbf{u} \).

Answer of exercise 2  
(a) Write down anticommutator and commutator of Pauli matrices using \( \delta_{ij} \) and \( \epsilon_{ijk} \) tensors

\[ [\sigma_i, \sigma_j]_+ = 2\delta_{ij} \]

\[ [\sigma_i, \sigma_j]_- = 2i\epsilon_{ijk} \sigma_k \]

(b) Express a product of Pauli matrices in terms of (a) results

\[ \sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k \]

(c) Using (b) express in terms of scalar and vector products of \( \mathbf{a}, \mathbf{b} \)

\[ (a \sigma)(b \sigma) = a \cdot b + i \sigma \cdot (a \times b) \]

and find traces over spins

\[ \text{Tr} \{ (a \sigma)(b \sigma) \} = 2(a \cdot b) \]

\[ \text{Tr} \{ \sigma (a \sigma)(b \sigma) \} = 2i(a \times b) \]